DAMPING OF FLEXURAL VIBRATIONS IN BEAMS

BY VISCOELASTIC MATERIALS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Jayant Saran Lal

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

October, 1966

DAMPING OF FLEXURAL VIBRATIONS IN BEAMS

BY VISCOELASTIC MATERIALS

Approved:

Chairman

Date approved by Chairman: May 4, 1966

In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

3/17/65
b

ACKNOWLEDGMENTS

The author of this work would like to express his most sincere thanks to all those persons who have contributed to the completion of this work, especially to Dr. J. R. Baumgarten, the thesis advisor, who never fell short in providing guidance and encouragement.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

Damping of flexural vibrations of a Timoshenko beam covered on one side by a viscoelastic material has been carried out by the classical theory of the damped oscillations. The analysis results in a fifth order partial differential equation of motion. Since a closed form solution does not exist, the approach taken was to assume a periodic solution that satisfied the equation of motion. The logarithmic decrement was found as a function of non-dimensional length of the beam and a non-dimensional viscosity parameter of the visco-elastic material which has been used as an adhesive layer to damp the flexural vibration of the beam. This approach has an advantage over the numerical analysis of the equation in the sense that the design of a structure subjected to random external forces having broad band characteristics requires knowledge of the "half-life period," rather than the actual wave form of the response. As a result of this study the designer is provided with the logarithmic decrement as a function of non-dimensional length and viscosity parameter.

The effects of rotary inertia and shear in the beam have also been considered in this study. These effects were first studied by Timoshenko on undamped freely vibrating beams. This investigation extends the existing work on the response of damped structures by the inclusion of rotary inertia and shear.

A relation was found between the logarithmic decrement of amplitude and non-dimensional viscosity parameter as a function of design

parameters. A Burroughs 220 digital computer program was written to give numerical values of this relationship.

CHAPTER I

INTRODUCTION

When mechanical elements must withstand vibration imposed in service, their natural frequency becomes a prime consideration. Frequency of vibration depends upon the mass and stiffness of the element. At natural frequencies the forces of inertia and spring forces become equal and this makes the amplitude of the response very large (infinite under ideal conditions). Such a phenomenon is termed resonance. In the event of the mechanical element being exposed to random forces with broad band characteristics, several resonances are excited simultaneously.

Techniques used to eliminate or reduce resonance or high amplitude of response, may be generalized as follows:

If the structure is a complex one, then the adjoining systems are designed with natural frequencies as far apart as the space and weight requirements permit. Obviously, the method can have very limited applications.

If the characteristics of the external forces are known in advance, then the mechanical elements are so designed that their natural frequencies do not fall within the frequency range to which they will be exposed.

The method which has found wide application may be called "damping the system." Damping of the system is normally carried out by using isolators. Isolators are frequency dependent and so cannot be used for damping of responses with broad band characteristics.

The study carried out has been on damping materials so organized that they become frequency independent. The vibrating beam has been used as the basis of all further development in this work. However, principles herein embodied are sufficiently general to be applied to any mechanical system.

Statement of the Problem

When a beam undergoes flexural vibrations, under the hypothetical conditions that there are no resisting forces, the beam will continue vibrating indefinitely. It has been observed that a layer of viscoelastic coating on the beam damps out the motion. This layer also undergoes vibrations and its extension and compression introduce shear forces. These shear forces cause dissipation of energy and thus tend to damp the vibration of the system.

Regardless of what mode of vibration is excited in the beam, the viscoelastic layer will have a response, and as long as the composite system moves, damping will result. This method of damping causes one to consider the effect of the shear of the vibrating beam itself. This problem was first studied by Timoshenko on an undamped free vibrating beam. This effect has also been considered in this work.

As the beam undergoes flexure, each transverse element of the beam like ab, Figure 1, rotates about its point on the neutral axis. This introduces another term called "rotary inertia" of the beam, in the equation of the motion.

The results obtained from theoretical treatment of the problem are

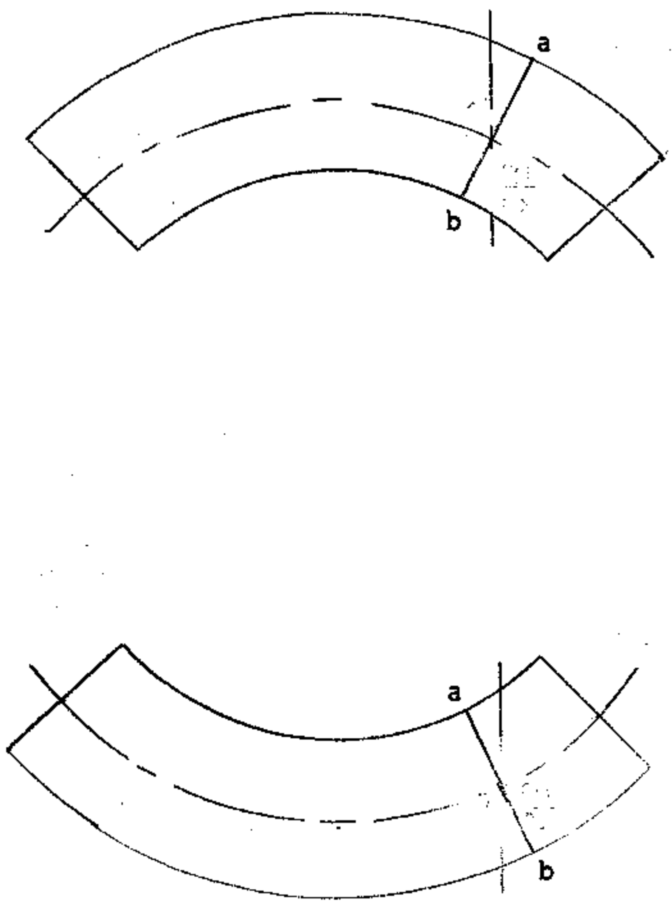


Figure 1. Beam Under Flexure.

verified by the experiments. For experimental purposes, rubber and wood were used as viscoelastic materials and sheet metal as the vibrating beam material.

Method of Analysis

The procedure followed in this study is one of analysis and experimental verification. First, the problem is treated analytically including such effects as shear in the beam and rotary inertia. Then the results of analysis are verified by experimental techniques.

Certain non-dimensional relationships are derived relating the logarithmic decrement to the beam length and viscosity parameter of the coating material. A digital computer program was written to determine the logarithmic decrement from this relationship. A family of curves resulting from this computer analysis is presented.

CHAPTER II

REVIEW OF THE LITERATURE

One of the major problems in the designing of high speed aeroplanes has been that of reducing vibrations. Lately automobile manufacturers have also been confronted with the same problem. This has resulted in extensive research and several papers have been published. Viscoelastic materials have been found to be very effective in damping flexural vibrations, when coated on the inside of the fuselage of an aeroplane or inside of the framework of an automobile. The papers which have been published on the subject can be classified into two distinct areas,

1. Integral or constrained-layer damping
2. Unconstrained-layer damping

Composite structures which are used for integral or constrained-layer type damping consist of two or more damping layers. A beam to which a single damping tape has been applied comprises a three-layer structure composed of

1. The beam to be damped
2. A viscoelastic layer, and
3. An outer-constraining layer, generally of metal foil.

In the case of N damping tapes, each damping tape consists of a layer of viscoelastic damping material and of an elastic constraining foil. A three layer composite beam has been illustrated in Figure 3.

In the first area, there are papers by Unger, Ross and Kervin (2),

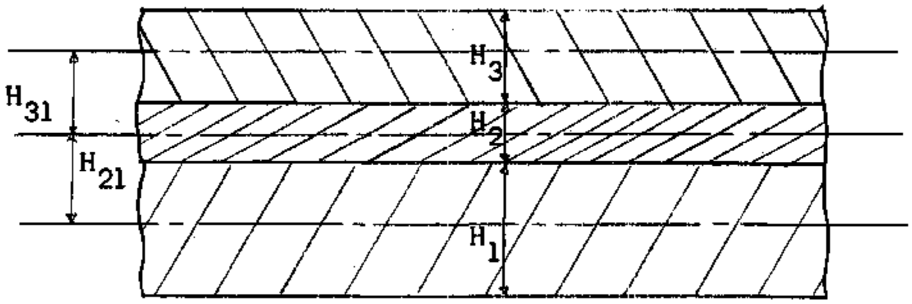


Figure 2. Element of Single-Layer Plate or Beam.

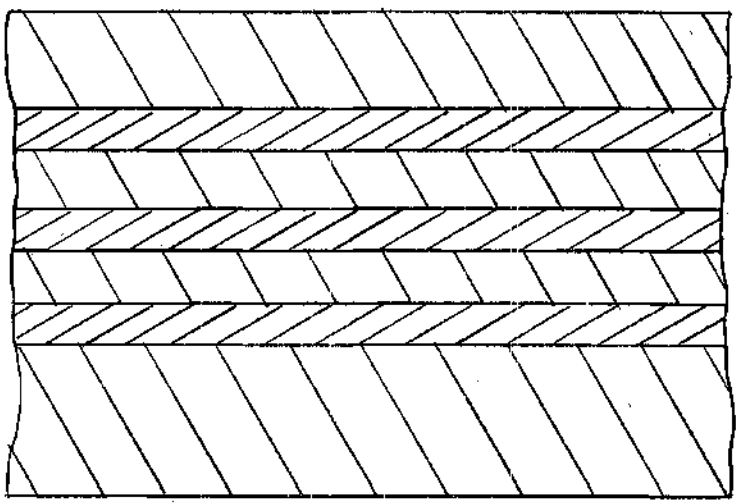


Figure 3. Element of Three-Layer Plate or Beam.

Whittier (6), and Thorn (12). The paper by Unger, Ross and Kervin (2) is by far the most comprehensive work in the field known to the author.

Unger, Ross and Kervin (2) have made a number of assumptions, the important ones being:

(1) The energy losses are assumed to be entirely attributable to the shear motion of the viscoelastic damping layers.

(2) Shear strains of the constraining layers are negligible.

(3) Extensional stresses of the viscoelastic layers are negligible.

The approach has been to find damping effectiveness η in terms of properties and dimensions of the base plate and the viscoelastic material. Damping effectiveness has been defined as the imaginary portion of the complex combined flexural rigidity B^*, that is $B^* = B(1 + i\eta)$. The real part B is associated with elastic behavior, whereas the imaginary part $B\eta$ is associated with hysteretic, or dissipative, behavior.

After making a simplifying assumption that the constraining layer is considerably less stiff than the base plate, they arrive at the relationship,

$$\eta = \frac{12\ Kh^2 \beta g}{1 + 2g + g^2(1 + \beta^2)}$$

where

$$g = \frac{G}{K_3 H_2 p^2}$$

G = Shear modulus of adhesive

$$G^* = G(1 + i\beta)$$

K_3 = Stiffness of constraining layer

$$p^2 = W_n\sqrt{\frac{m}{B}} = (\text{wave number})^2$$

$$h = H_{31}/H_1$$

$$K = K_3/K_1 \ (\ll 1)$$

In spite of the number of simplifying assumptions, the resulting expression is rather complicated. In addition

(1) There is no means of finding β

(2) Loss factor (η) depends upon frequency

(3) Loss factor (η) depends upon temperature

(4) The analysis is not a general one and can not be extended readily.

Thorn's (12) work has also been based on the concept of loss factor and so suffers from the same drawbacks as the analysis of the problem by Unger, Ross, and Kervin (2).

Thorn's equation for the composite loss factor of any symmetrical damped laminate is

$$\eta = \frac{2\beta Y g}{1 + (2 + Y)2g + (1 + \beta^2)(1 + Y)4g^2}$$

where

β = Core material loss factor

$$Y = 3(1 + h)^2$$

$$h = H_2/H_1$$

H_1 = Plate thickness

H_2 = Damping layer thickness

$$g = \frac{G'}{K_1 H_2 W} \left(\frac{B}{m}\right)^{1/2}$$

G' = Dynamic elastic shear modulus

$$K_1 = E_1 H_1$$

B = Bending stiffness lb-in

m = mass per unit area lb-sec^2 per in^4

$$B = \frac{K_1 H_1^2}{6} + \frac{K_1 g (H_1 + H_2)^2}{1 + 2g}$$

The author is not familiar with any work which might have been done in the area of constrained-layer type of damping which is either based on the classical theory of flexural vibrations or even gives a simple expression which can be used readily for design purpose. All other work which the author has come across in this field is rather vague.

In the unconstrained-layer type of damping, there have been papers by Plass (7), Lienard (13) and Oberst (14).

Oberst has used theory of flexural vibration of beam to derive an expression for the loss factor.

$$\frac{\eta}{\eta_2} = \frac{a\xi}{1 + a\xi} \cdot \frac{3 + 6\xi + 4\xi^2 + 2a\xi^3 + a^2\xi^4}{1 + 2a(2\xi + 3\xi^2 + 2\xi^3) + a^2\xi^4}$$

where

$a = E_2/E_1$

E_1 = Young's Modulus of the material of beam

E_2 = Young's Modulus of the coating material

$\xi = d_2/d_1$

d_1 = thickness of the beam

d_2 = thickness of the coating material

η_2 = loss factor of the coating material

Again, there is no means of finding loss factor of the coating material.

Lienard (13) has used energy methods for finding damping effects. He has found expressions for energy loss per cycle of the homogeneous beam and also of the coated beam. The difference in the energy loss per cycle in the two cases gives an idea of the damping effect.

Energy loss per cycle is given by:

$$\Delta W_1/W = [5.2 \left(\frac{e}{L}\right)^3 \frac{1}{m}] \frac{\mu}{N} \qquad (1)$$

where

ΔW_1 = energy loss

2L = length of the beam

m = mass of the bar per unit width

e = thickness of the beam

μ = friction coefficient (viscous)

N = frequency of free vibrations

and

$$W = \pi^2 m\ A^2 N^2 \quad \text{for principal mode.}$$

If we consider a non-linear viscosity in which the force

$$F = v\ \frac{e/L}{\frac{de}{dt}}$$

$$\frac{\Delta W_1}{W} = \left[500\ \left(\frac{e}{L}\right)^4 \frac{1}{m}\right] a(v/N) \tag{2}$$

where

$a = \frac{A}{2L}$

A = amplitude

2L = length

In the case of a solid friction

$$\frac{\Delta W_3}{W} = \left[0.33\ \left(\frac{e}{L}\right)^2 \left(\frac{1}{m}\right)\right] \frac{1}{a}\ (f/N^2) \tag{3}$$

Three ratios have been considered for a mixed metal-plastic beam. If

e = thickness of coating

ρ_1 = density of coating material

E_1 = modulus of elasticity

The same symbols without indices are used for the beams

$$\frac{e_1}{e} = h_1 \quad \frac{\rho_1}{\rho} = d \,. \quad \frac{E_1}{E} = Q$$

In the case of viscous friction, corresponding to equation (1)

$$\frac{\Delta W \rho_1}{W} = [5.2 \left(\frac{e}{L}\right)^3 \left(\frac{1}{mt}\right)(13h^3 \mu_1)N] \tag{4}$$

where

$$mt = m(1 + hd)$$

$$W = 10 m_t A^2 N^2$$

In the case of non linear viscous friction, corresponding to Equation (2)

$$\frac{\Delta W_{\rho 1}}{W} = [500 \left(\frac{e}{L}\right)^4 \left(\frac{1}{mt}\right)] \, 4h(h + 1)(1 + 2h + 2h^2) a v_1 / N(5) \tag{5}$$

In the case of solid friction, corresponding to Equation (3)

$$\frac{\Delta W_{\rho 3}}{W} = [0.33 \left(\frac{e}{L}\right)^2 \left(\frac{1}{mt}\right)] \, 2h(h + 1) \; f_1 / aN^2 \tag{6}$$

From Eqs. (1) through (6), one can find the effect of damping.

Plass (7) extended the classical theory of flexural vibration of a beam to incorporate the damping effects of viscoelastic materials. In the analysis a few mistakes were found. Plass (7) has made two assumptions which the author finds difficult to comprehend.

First, the viscosity parameter β_c of the viscoelastic material has been defined as

$$\beta_c = \frac{(E_c)a - E_c}{\eta_c}$$

where

$(E_c)a$ is adiabatic modulus of the coating

E_c is isothermal modulus of the coating

η_c is viscosity parameter of the coating

As $(E_c)a$, E_c and η_c are all independent of the physical dimensions of the coating, it implies that the damping effect of the coating is quite independent of the frequency of vibration of the beam, and so of the coating.

Second, C^*, which has been defined as

$$C^* = \frac{E_b + (E_c)a \dfrac{I_c}{I_b}}{E_b + E_c \dfrac{I_c}{I_b}}$$

where index b stands for beam and c for the coating, has been assumed to be 1.1. C^* is a function of I_c/I_b and hence if the thickness ratio of the coating and the beam changes, C^* cannot remain constant.

These two difficulties have been overcome as follows: Nolle (4), instead of using adiabatic and isothermal modulii of elasticity, has used the quantities which are real and imaginary parts E_1 and E_2 of the complex dynamical modulus E^*.

$$E^* = E_1 - iE_2$$

The quantity E_1 is the ordinary or isothermal Young's modulus in the case of a material which has no viscous losses, and is therefore a measure of the magnitude of the elastic force and is in phase with displacement.

E_2 is a measure of the viscous force (magnitude) which is out of phase with the displacement by $\frac{\pi}{2}$ radians. It has been shown that

$$E_2 = \eta_c \omega_c$$

where ω_c is frequency in radians per second of viscoelastic material.

Thus,

$$\beta_c = \left| \frac{E_1 - i\eta_c\omega_c - E_1}{\eta_c} \right|$$

$$= \left| \omega_c \right|$$

Hence, β_c is equal to ω_c.

The value of C^* in this investigation is assumed as follows:

Rubber and wood were used as viscoelastic materials. Rubber was found to be reasonably effective as a damping material, where the thickness ratio was 18.

i.e.

$$\frac{d}{D} = 18$$

where

d = thickness of rubber coating

D = thickness of the beam

Using this ratio as 18, and other properties of rubber as found experimentally, C^* came to be 1.025. This value of C^* was used to plot the family of curves and also to find the implied thickness ratio for other combinations.

CHAPTER III

CONCEPT OF ROTARY INERTIA AND EFFECT OF SHEAR IN BEAM

Effect of Rotary Inertia

Each element which is transverse to the neutral axis of a beam undergoing flexural vibrations like element ab in Figure 4, rotates also about point C on the neutral axis. The variable angle of rotation is equal to the slope of the deflection curve expressed by

$$\frac{\partial w}{\partial x}$$

and the corresponding angular velocity and angular acceleration will be given by

$$\frac{\partial^2 w}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^3 w}{\partial x \partial t^2} \quad \text{respectively.}$$

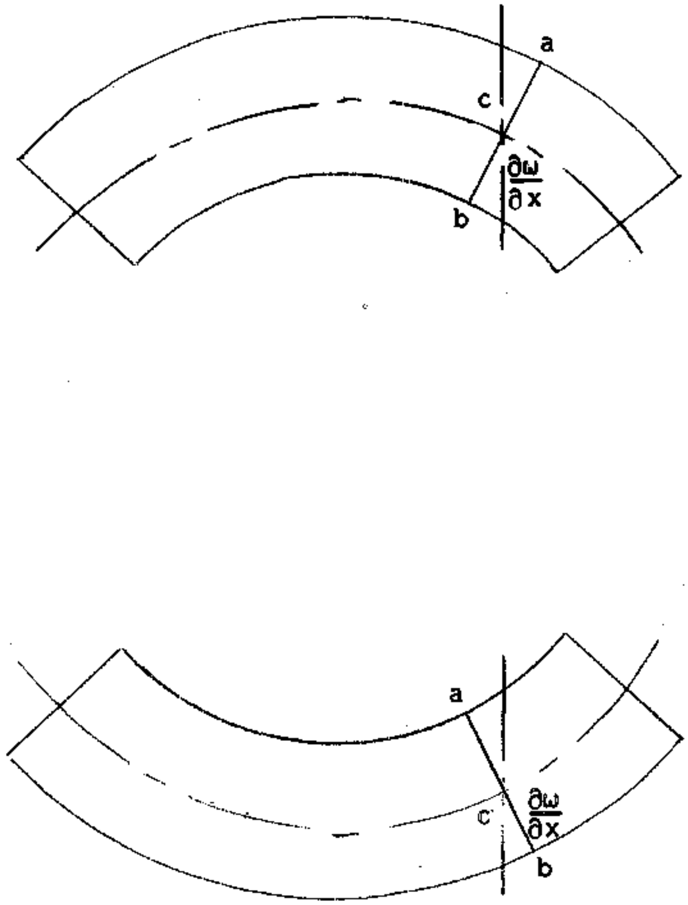
Therefore the inertia torques about the axis through its centre of gravity and perpendicular to the xw plane will be

$$-I_b \rho_b \frac{\partial^3 w}{\partial x \partial t^2} \cdot dx$$

By summing moments,

$$\frac{\partial M}{\partial x} \cdot dx = Q\, dx - I_b \rho_b \frac{\partial^3 w}{\partial x \partial t^2} \cdot dx$$

or



Beam 4. Beam Under Flexure.

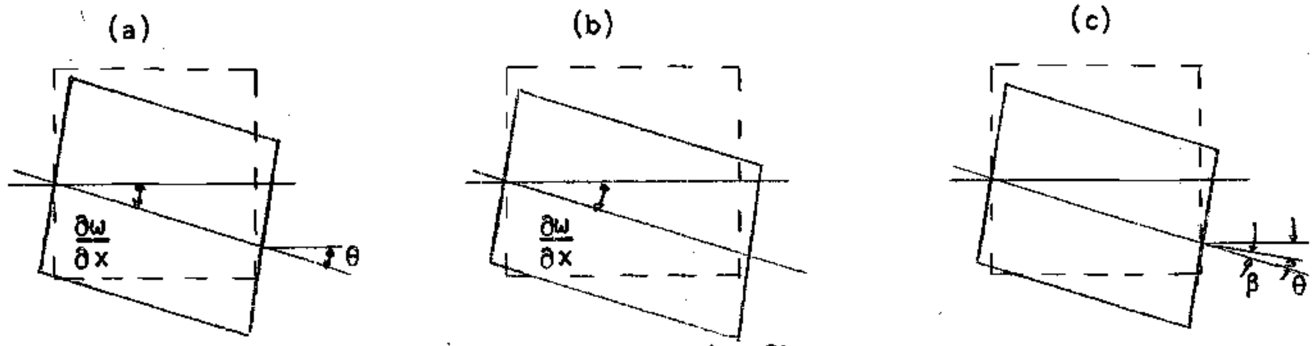


Figure 5. Beam Under Shear.

$$\frac{\partial M}{\partial x} = Q - I_b \rho_b \frac{\partial^3 w}{\partial x \partial t^2} \qquad (1)$$

Effect of Shear in the Beam

Figure 5(a) shows section of a beam which has been subjected to deflection, but no shear has been induced. In this case,

$$\theta = \frac{\partial w}{\partial x}$$

Figures 5(b) and (c) show the same section of the beam with shear induced. In this case,

$$\frac{\partial w}{\partial x} = \theta + \beta \qquad (2)$$

where β is the angle of shear at the neutral axis in the same section.

Then,

$$M = - E_b I_b \frac{\partial \theta}{\partial x} \qquad (3)$$

$$Q = K' \beta A_b G_b$$

K' is a constant depending upon the cross section[15]

$$\therefore\ Q = K'\left(\frac{\partial w}{\partial x} - \theta\right) A_b G_b$$

Substituting this value in equation (1),

$$E_b I_b \frac{\partial^2 \theta}{\partial x^2} + K'\left(\frac{\partial w}{\partial x} - \theta\right) A_b G_b - I_b \rho_b \frac{\partial^2 \theta}{\partial t^2} = 0 \qquad (4)$$

For equilibrium in the transverse direction,

$$\frac{\partial Q}{\partial x} dx = A_b \rho_b \frac{\partial^2 w}{\partial t^2} \cdot dx$$

or

$$A_b \rho_b \frac{\partial^2 w}{\partial t^2} - K' \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) A_b G_b = 0 \qquad (5)$$

Eliminating θ from (4) and (5) we get

$$E_b I_b \frac{\partial^4 w}{\partial x^4} + A_b \rho_b \frac{\partial^4 w}{\partial t^2} - I_b \rho_b \frac{\partial^2 w}{\partial x^2 \partial t^2} - \frac{I_b \rho}{K' G_b} \left(E_b \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho \frac{\partial^4 w}{\partial t^4}\right) = 0$$

In this equation, the third term is due to rotary inertia and fourth due to the shear in the beam.

Rearranging the terms in the equation,

$$\frac{\partial^4 w}{\partial x^4} - \left[\frac{\rho_b}{E_b} + \frac{\rho_b}{K' G_b}\right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho_b}{E_b} \cdot \frac{\rho_b}{K' G_b} \cdot \frac{\partial^4 w}{\partial t^4} + \frac{\rho_b}{E_b} \frac{A_b}{I_b} \cdot \frac{\partial^2 w}{\partial t^2} = 0$$

Writing this equation in terms of longitudinal velocity of the stress waves in the beam,

$$\frac{\partial^4 w}{\partial x^4} - \left[\frac{1}{C_o^2} + \frac{1}{C_\varphi^2}\right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{C_o^2 C_\varphi^2} \frac{\partial^4 w}{\partial t^4} + \frac{1}{C_o^2 R^2} \frac{\partial^2 w}{\partial t^2} = 0$$

This is the Timoshenko beam equation and the same approach is used to incorporate the effects of rotary inertia and shear in the beam which has been coated with a viscoelastic material.

CHAPTER IV

BEHAVIOR OF VISCOELASTIC MATERIALS

Classical theory of elasticity assumes that the stress-strain relations are linear and independent of time. A number of bodies exhibit certain deviations from the above assumption and the stress-strain relations contain a new variable - the time.

Time variations of the mechanical properties of the medium are accounted for by regarding the medium viscoelastic, i.e., combining two media, one which is perfectly elastic, while the second has the properties of a viscous fluid: elastic phenomema are governed by Hooke's law and the viscous phenomena by Newton's law.

Behavior of viscoelastic bodies can be explained by means of models consisting of a system of springs and dashpots (9). The springs describe the elastic properties, and the dashpots the viscous properties. The system of springs and dashpots can be so chosen that the system fits as closely as possible to the experimental curves of the viscoelastic materials.

In this study, the amplitudes of vibration are assumed to be small. Therefore the behavior of viscoelastic material may be considered to be linear. This also implies that Boltzmann's superposition principle will hold. This principle states that if a stress cycle $\sigma_1(t)$ produces the deformation $\varepsilon_1(t)$ and the stress cycle $\sigma_2(t)$ the deformation $\varepsilon_2(t)$, then the sum of the cycle

$$\sigma_1(t) + \sigma_2(t)$$

produces the deformation

$$\varepsilon_1(t) + \varepsilon_2(t) .$$

The behavior of viscoelastic materials is best explained by a Kelvin-Maxwell combination model as shown in Figure 6(a). When a Kelvin model, shown in Figure 6(b), is subjected to a slowly applied load, no resistance is offered by the dashpot as sufficient time is allowed for fluid to flow past the plunger. If the load is applied suddenly, the dashpot makes the model behave as a rigid body. In the Maxwell model, shown in Figure 6(c), slowly applied loads result in indefinite elongation.

The combination model neither behaves as a rigid body to suddenly applied loads, nor gives unlimited deformation when exposed to slowly applied loads. In this treatment it is assumed that springs and dashpots are so designed that load-stress and deformation-strain becomes identically equal.

In the Kelvin model, the extensions of the spring ε_S and of the dashpots ε_D are equal and the stress is composed of two parts σ_S and σ_D, i.e.

$$\sigma = \sigma_D + \sigma_S$$

$$\sigma_S = E\ \varepsilon_S$$

$$\sigma_D = \eta \frac{\partial \varepsilon_D}{\partial t}$$

$$\sigma = \eta \frac{\partial \varepsilon_D}{\partial t} + E\ \varepsilon_S$$

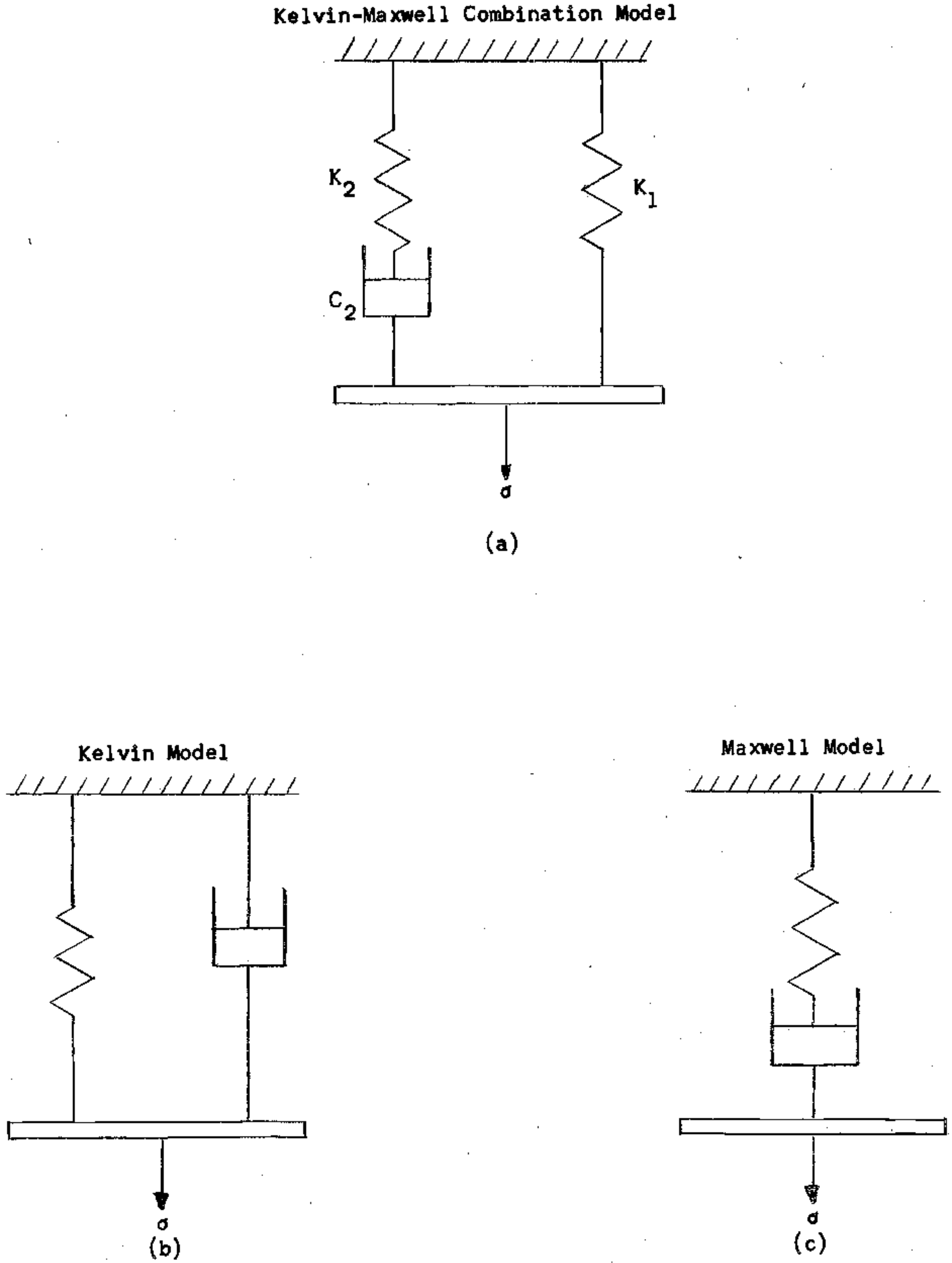


Figure 6. Model of Viscoelastic Material

For the Maxwell model

$$\varepsilon = \varepsilon_s + \varepsilon_D$$

$$\sigma = E\,\varepsilon_s$$

$$\sigma = \eta\,\frac{\partial \varepsilon_D}{\partial t}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{E}\frac{\partial \sigma}{\partial t} + \frac{1}{\eta}\,\sigma$$

$$\sigma = \eta\,\frac{\partial \varepsilon}{\partial t} - \frac{\eta}{E}\frac{\partial \sigma}{\partial t}\,.$$

or

Proceeding along similar lines for the model shown in Figure 6(a)

$$\sigma = -\frac{C_2}{K_2}\frac{\partial \sigma}{\partial t} + C_2 \cdot \frac{K_1 + K_2}{K_2}\frac{\partial \varepsilon}{\partial t} + K_1\varepsilon$$

Arranging terms,

$$(K_1 + K_2)\,\frac{\partial \varepsilon}{\partial t} - \frac{\partial \sigma}{\partial t} = \frac{K_2}{C_2}\,(\sigma - K_1\varepsilon)$$

The corresponding stress-strain-time relations for simple axial stress conditions in the viscoelastic materials corresponding to the model shown in Figure 6(a) is:

$$(E_c)_A\,\frac{\partial \varepsilon_c}{\partial t} - \frac{\partial \sigma_c}{\partial t} = \frac{(E_c)_A - E_c}{\eta}\,(\sigma_c - E\varepsilon)$$

where η is the viscosity parameter, E_A is the dynamic modulus of elasticity and E is the isothermal modulus of elasticity. Defining β_c as

$$\beta_c = \frac{(E_c)_A - E_c}{\eta}$$

the equation becomes

$$(E_c)_A \frac{\partial \varepsilon_c}{\partial t} - \frac{\partial \sigma_c}{\partial t} = \beta_c(\sigma_c - E_c \varepsilon)$$

CHAPTER V

DEVELOPMENT OF THE EQUATION OF MOTION

As shown in Figure 7, the angle of shear β of two transverse faces of an element of beam can be represented by

$$\beta = \frac{KQ}{AG}$$

$$= \frac{K}{AG} \frac{\partial M}{\partial x}$$

K is a constant depending on the shape of the cross section only.

The curvature due to shear will depend upon the rate of change of angle of shear along the length.

Therefore

$$E_b I_b \frac{\partial^2 w}{\partial x^2} = - M_b + \frac{K E_b I_b}{A_b G} \frac{\partial^2 M_b}{\partial x^2} \tag{1}$$

Substituting

$$\sigma_c = - M_c \frac{Z}{I_c}$$

and

$$\varepsilon_c = + Z \frac{\partial^2 w}{\partial x^2}$$

into equation (1), Chapter IV,

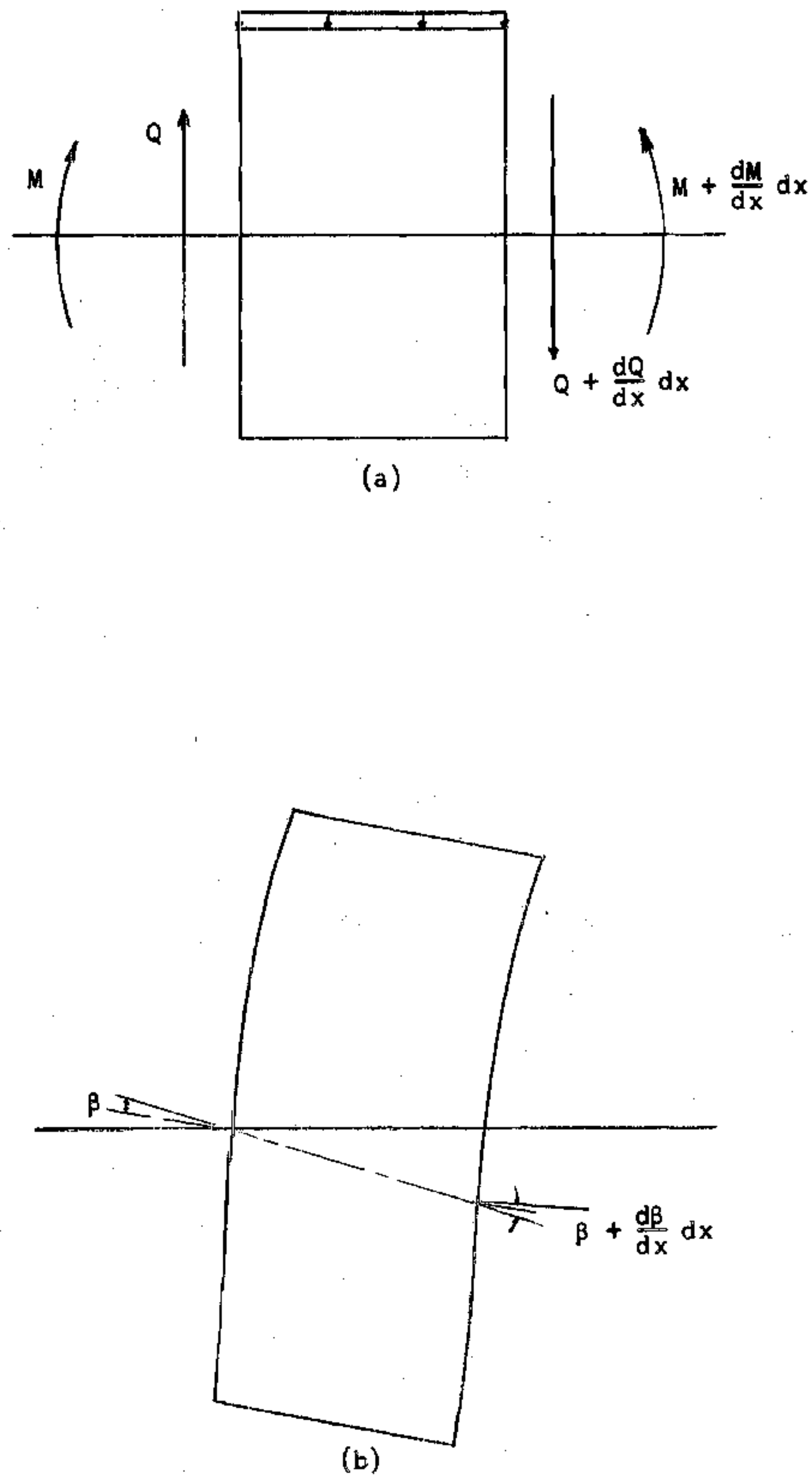


Figure 7. Relative Angle of Sliding β.

$$(E_c)_A \frac{\partial \varepsilon_c}{\partial t} - \frac{\partial \sigma_c}{\partial t} = \beta_c(\sigma_c - E_c\varepsilon_c)$$

yields

$$(E_c)_A I_c \frac{\partial^3 w}{\partial t \partial x^2} + \frac{\partial M_c}{\partial t} = -\beta_c(M_c + E_c I_c \frac{\partial^2 w}{\partial x^2}) \tag{2}$$

The total bending moment of the composite structure is taken as the sum of moments in the beam and the coating. Equation (1) is differentiated and combined with equation (2)

$$\left[E_b \frac{I_b}{I_b} + (E_c)_A \frac{I_c}{I_b}\right] I_b \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\partial}{\partial t}(M_b + M_c)$$

$$= -\beta_c \left[(M_b + M_c) + (E_b \frac{I_b}{I_b} + E_c \frac{I_c}{I_b}) I_b \frac{\partial^2 w}{\partial x^2}\right]$$

$$+ \frac{KE_b I_b}{A_b G} \cdot \frac{\partial^3 M_b}{\partial x^2 \partial t} + \frac{KE_b I_b}{A_b G} \beta_c \cdot \frac{\partial^2 M_b}{\partial x^2}$$

or

$$E_a^* I_b \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\partial M}{\partial t} = -\beta_c \left[M + E^* I_b \frac{\partial^2 w}{\partial x^2}\right] + \frac{KE_b I_b}{A_b G} \cdot \frac{\partial^3 M_b}{\partial x^2 \partial t}$$

$$+ \frac{KE_b I_b}{A_b G} \beta_c \cdot \frac{\partial^2 M_b}{\partial x^2}$$

The deflection caused by shear in the beam is negligible as compared to the deflection caused by the moment. Thus without introducing any appreciable error, it may be assumed that

$$E_b I_b \frac{\partial^2 M_b}{\partial x^2} = E_a^* I_b \frac{\partial^2 M}{\partial x^2}$$

Differentiating twice with respect to x,

$$E_a^* I_b \frac{\partial^5 w}{\partial x^4 \partial t} + \frac{\partial^3 M}{\partial x^2 \partial t} = -\beta_c \left[\frac{\partial^2 M}{\partial x^2} + E^* I_b \frac{\partial^4 w}{\partial x^4} \right]$$

$$+ \frac{KE_a^* I_b}{A_b G} \cdot \frac{\partial^5 M}{\partial x^4 \partial t} + \frac{KE_a^* I_b}{A_b G} \beta_c \frac{\partial^4 M}{\partial x^4} \qquad (3)$$

From Chapter III, equation (1),

$$\frac{\partial M}{\partial x} - Q + I \rho \frac{\partial^3 w}{\partial x \partial t^2} = 0 \qquad (4)$$

Considering the equilibrium of an element in the transverse direction.

$$\frac{\partial Q}{\partial x} - \rho A_b \frac{\partial^2 w}{\partial t^2} = 0 \qquad (5)$$

where

$$\rho A_b = \rho_b A_b + \rho_c A_c$$

Equations (4) and (5) give

$$\frac{\partial^2 M}{\partial x^2} = \rho \left[A_b \frac{\partial^2 w}{\partial t^2} - I_b \frac{\partial^4 w}{\partial x^2 \partial t^2} \right]$$

and

$$\frac{\partial^4 M}{\partial x^4} = \rho \left[A_b \frac{\partial^4 w}{\partial x^2 \partial t^2} - I_b \frac{\partial^6 w}{\partial x^4 \partial t^2} \right]$$

Substituting values of $\frac{\partial^2 M}{\partial x^2}$ and $\frac{\partial^4 M}{\partial x^4}$ into equation (3), and arranging terms,

$$E_a^* I_b \frac{\partial^5 w}{\partial x^4 \partial t} - \rho \left[- A_b \frac{\partial^3 w}{\partial t^3} + I_b \frac{\partial^5 w}{\partial x^2 \partial t^3} \right]$$

$$= - \beta_c \left[\rho A_b \frac{\partial^2 w}{\partial t^2} - I_b \rho \frac{\partial^4 w}{\partial x^2 \partial t^2} + E^* I_b \frac{\partial^4 w}{\partial x^4} \right]$$

$$+ \beta_c \left[A_b \frac{\partial^4 w}{\partial x^2 \partial t^2} - I_b \frac{\partial^6 w}{\partial x^4 \partial t^2} \right] \frac{KE_a^* I_b}{A_b G} \cdot \rho$$

$$+ \left[A_b \frac{\partial^5 w}{\partial x^2 \partial t^2} - I_b \frac{\partial^7 w}{\partial x^4 \partial t^3} \right] \frac{KE_a^* I_b}{A_b G} \rho$$

or

$$-I_b \rho \frac{\partial^5 w}{\partial x^2 \partial t^3} + \rho A_b \frac{\partial^3 w}{\partial t^3} - I_b \beta_c \rho \frac{\partial^4 w}{\partial x^2 \partial t^2} + \beta_c \rho A_b \frac{\partial^2 w}{\partial t^2}$$

$$- \frac{KE_a^* I_b}{A_b G} \rho \beta_c \left[\frac{\partial^4 w}{\partial x^2 \partial t^2} A_b - I_b \frac{\partial^6 w}{\partial x^4 \partial t^2} \right] + E_a^* I_b \frac{\partial^5 w}{\partial x^4 \partial t}$$

$$+ E^* I_b \beta_c \frac{\partial^4 w}{\partial x^4} - \frac{KE_a^* I_b}{A_b G} \rho \left[- A_b \frac{\partial^5 w}{\partial x^2 \partial t^3} + I_b \frac{\partial^7 w}{\partial x^4 \partial t^3} \right] = 0 \qquad (6)$$

Equation (6) is the more exact equation of the flexural vibration of the beam. This is a differential equation of 7th order. The solution of this equation will give nearly perfect response of the beam.

In this study simplifications are made to reduce the order of the equation to five. This is done as follows:

From the equilibrium of an element in the vertical direction,

$$\frac{\partial Q}{\partial x} \cdot dx = \rho A_b \frac{\partial^2 w}{\partial t^2} \cdot dx$$

$$\rho A_b \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 M}{\partial x^2}$$

$$= E_a^* I_b \frac{\partial^4 w}{\partial x^4}$$

$$\therefore I_b \frac{\partial^6 w}{\partial x^4 \partial t^2} = \frac{\rho A_b}{E_a^*} \cdot \frac{\partial^4 w}{\partial t^4}$$

and

$$I_b \frac{\partial^7 w}{\partial x^4 \partial t^3} = \frac{\rho A_b}{E_a^*} \cdot \frac{\partial^5 w}{\partial t^5}$$

Substituting these values into equation (6) and dividing the equation by ρA_b so that the equation may be written in terms of non-dimensional quantities:

$$- \frac{I_b}{A_b} \frac{\partial^5 w}{\partial x^2 \partial t^3} + \frac{\partial^3 w}{\partial t^3} - \frac{I_b}{A_b} \beta_c \frac{\partial^4 w}{\partial x^2 \partial t^2} + \beta_c \frac{\partial^2 w}{\partial t^2} + \frac{E_a^* I_b}{\rho A_b} \cdot \frac{\partial^5 w}{\partial x^4 \partial t}$$

$$+ K \frac{E_a^* I_b}{\rho A_b} \cdot \frac{\rho \beta_c}{G} \left[- \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho}{E_a^*} \frac{\partial^4 w}{\partial t^4} \right] + E^* \frac{I_b \beta_c}{\rho A_b} \frac{\partial^4 w}{\partial x^4}$$

$$+ K \frac{E_a^* I_b}{\rho A_b} \cdot \frac{\rho}{G} \left[- \frac{\partial^5 w}{\partial x^2 \partial t^3} + \frac{\rho}{E_a^*} \cdot \frac{\partial^5 w}{\partial t^5} \right] = 0$$

If D is the thickness of the beam, then the equation can be modified to

$$-\left[\frac{I_b}{A_bD^2}\right]\frac{\partial^5\left(\frac{w}{D}\right)}{\partial\left(\frac{w}{D}\right)^2\partial\left(\frac{tCo^*}{D}\right)^3}+\frac{\partial^3\left(\frac{w}{D}\right)}{\partial\left(\frac{tCo^*}{D}\right)^3}-\left[\frac{I_b}{A_bD^2}\right]\left[\frac{D\beta_c}{Co^*}\right]\frac{\partial^4\left(\frac{w}{D}\right)}{\partial\left(\frac{x}{D}\right)^2\partial\left(\frac{tCo^*}{D}\right)^2}$$

$$+\left[\frac{D\beta_c}{Co^*}\right]\frac{\partial^2\left(\frac{w}{D}\right)}{\partial(tCo^*)^2}+\frac{E_a}{\rho}\circ\frac{1}{Co^{*2}}\left[\frac{I_b}{A_bD^2}\right]\circ\frac{\partial^5\left(\frac{w}{D}\right)}{\partial\left(\frac{x}{D}\right)^4\partial\left(\frac{tCo^*}{D}\right)}$$

$$+\frac{E^*}{\rho Co^*}\left[\frac{I_b}{A_bD^2}\right]\circ\left[\frac{D\beta_c}{Co^*}\right]\circ\frac{\partial^4\left(\frac{w}{D}\right)}{\partial\left(\frac{x}{D}\right)^4}$$

$$+K\left[\frac{E_a^*}{\rho}\right]\circ\left[\frac{I_b}{A_bD^2}\right]\circ\left(\frac{\rho}{G}\right)\frac{D\beta_c}{Co^*}\left[-\frac{\partial^4\left(\frac{w}{D}\right)}{\partial\left(\frac{x}{D}\right)^2\partial\left(\frac{tCo^*}{D}\right)^2}+\frac{\rho}{E_a^*}\frac{Co^{*2}\partial^4\left(\frac{w}{D}\right)}{\partial\left(\frac{tCo^*}{D}\right)^4}\right]$$

$$+K\left[\frac{E_a^*}{\rho}\right]\circ\left[\frac{I_b}{A_bD^2}\right]\circ\left(\frac{\rho}{G}\right)\left[-\frac{\partial^5\left(\frac{w}{D}\right)}{\partial\left(\frac{x}{D}\right)^2\partial\left(\frac{tCo^*}{D}\right)^2}+\frac{\rho}{E_a^*}\frac{Co^{*2}\partial^5\left(\frac{w}{D}\right)}{\partial\left(\frac{tCo^*}{D}\right)^5}\right]=0$$

where $Co^* = \sqrt{\frac{E^*}{\rho}}$

the equation becomes

$$-K'^2\frac{\partial^5 w'}{\partial x'^2\partial t'^3}+\frac{\partial^3 w'}{\partial t'^3}-K'^2\beta_c'\frac{\partial^4 w'}{\partial x'^2\partial t'^2}+\beta_c'\frac{\partial^2 w'}{\partial t'^2}$$

$$+\frac{E_a^*}{E^*}K'^2\frac{\partial^5 w'}{\partial x'^4\partial t'}+K'^2\beta_c'\frac{\partial^4 w'}{\partial x'^4}+K\frac{E_a^*}{G}K'^2\beta_c'\left[-\frac{\partial^4 w'}{\partial x'^2\partial t'^2}\right.$$

$$\left.+\frac{E^*}{E_a^*}\circ\frac{\partial^4 w'}{\partial t'^4}\right]+K\circ\frac{E_a^*}{G}\circ K'^2\left[-\frac{\partial^5 w'}{\partial x'^2\partial t'^3}+\frac{E^*}{E_a^*}\circ\frac{\partial^5 w'}{\partial t'^5}\right]=0 \quad (8)$$

where primes represent nondimensional quantities.

The solution of equation (8) is of the form

$$w(x', t') = W(x') \cdot e^{\alpha' t'}$$

Substitution into the equation yields

$$\alpha'^5 \left[K \cdot K'^2 \frac{E_a^*}{G} \cdot \frac{E^*}{E_a^*} \cdot W \right] + \alpha'^4 \left[K \cdot K'^2 \cdot \frac{E_a^*}{G} \frac{E^*}{E_a^*} \cdot \beta_c' \cdot W \right]$$

$$+ \alpha'^3 \left[W - K'^2 W^{ii} - \frac{KE_a^*}{G} \cdot K'^2 W^{ii} \right] + \alpha'^2 \left[W - K'^2 W^{ii} \right.$$

$$\left. - \frac{KE_a^*}{G} \cdot K'^2 W^{ii} \right] \beta_c' + \alpha' \left[\frac{E_a}{E^*} K'^2 W^{iv} \right] + \left[K'^2 \beta_c' W^{iv} \right] = 0 \qquad (9)$$

If the equation of the wave form at any instance can be known, then this equation can be used to evaluate the logarithmic decrement or the half life period as a function of β_c, the viscosity parameter of the viscoelastic material. The wave form can be found from end conditions as follows:

Rearranging terms of equation (9),

$$W^{iv} \left[K'^2 (\beta_c + \alpha' \frac{E_a^*}{E^*}) \right] - W^{ii} \left[(\alpha' + \beta_c') K'^2 (1 + \frac{KE_a^*}{G}) \alpha'^2 \right]$$

$$+ W \left[\alpha'^4 (\alpha' + \beta_c') K \cdot K'^2 \cdot \frac{E^*}{G} \right] = 0$$

For convenience sake, writing the equation in the form

$$W^{iv} - 2b^2 W^{ii} + c^2 W = 0$$

where

$$2b^2 = \frac{(\alpha' + \beta_c')(1 + \frac{KE_a^*}{G})\alpha'^2}{\beta_c' + \alpha' \frac{E_a^*}{E^*}}$$

and

$$c^2 = \frac{(\alpha' + \beta_c')K \circ K'^2 \circ \frac{E^*}{G}\alpha'^4}{\beta_c' + \alpha' \frac{E_a^*}{E^*}}$$

For a simply supported beam, the boundary conditions are:

$$W(0) = W(1) = W^{ii}(0) = W^{ii}(1) = 0$$

To solve the equation, finite sine transforms are used. The equation becomes:

$$\left[(\frac{n\pi D}{l})^4 + 2(\frac{n\pi D}{l})^2 + c^2\right]\int_0^{\ell/D} W \sin \frac{n\pi D}{l} x' \, dx' = A \qquad (10)$$

From Fourier Series,

$$W(x',t') = \frac{2Ae^{\alpha' t'}}{\ell} \sum_{n=1}^{\infty} \frac{(\beta_c' + \alpha' \frac{E_a^*}{E^*}) \sin\left[n(\frac{\pi D}{\ell}) \cdot x'\right]}{(\frac{n\pi D}{\ell})^4 + (\frac{n\pi D}{\ell})^2(\alpha' + \beta_c')(1 + \frac{KE_a^*}{G})\alpha'^2 + (\alpha' + \beta_c')K \circ K'^2 \frac{E^*}{G}\alpha'^4}$$

(11)

If α' is found, then the exact solution of the beam may be obtained.

For $n = 2, 4, 6, \ldots,$ at midspan the numerator is zero. The denominator of the equation (11) has the term $\alpha_n^{\ 4}$.

$$\alpha_1^{\ 4} = \left(\frac{\pi D}{l}\right)^4$$

$$\alpha_3^{\ 4} = 81 \left(\frac{\pi D}{l}\right)^4$$

$$\alpha_5^{\ 4} = 3125 \left(\frac{\pi D}{l}\right)^4$$

Hence the series converges very rapidly.

For the purposes of experimental verification, only the first term of the series may be considered. Writing the equation in the form

$$w'(x', t') = B \sin L' x' e^{\alpha' t'}$$

where L' is some function of length, and for a simply supported beam,

$$L' = \frac{\pi D}{l}$$

Substituting

$$L'' = K'L' \, 10^5$$

$$\beta_c'' = \beta_c' \, 10^5$$

and

$$\alpha'' = \alpha' \, 10^5$$

$$\left(KK'^2\frac{E^*}{G}\right)\alpha''^5\,10^{-10} + \beta_c''\left(KK'^2\frac{E^*}{G}\right)\alpha''^4\,10^{-10} +$$

$$\left(1 + K'\,10^{-5} + KK'\frac{E_a^*}{G}\,10^{-5}\right)\alpha''^3 + \beta_c''\left(1 + K'\,10^{-5} + KK'\frac{E_a^*}{G}\,10^{-5}\right)\alpha''^2 +$$

$$\left(\frac{E_a^*}{E^*}L''^2\right)\alpha'' + \beta_c''\left(L''^2\right) = 0 \qquad (12)$$

The solution of this equation (α'') is the damping coefficient, and so the logarithmic decrement. The theory of the computer program is given in Chapter VI for the solution of this equation.

CHAPTER VI

THEORY OF THE COMPUTER SOLUTION

The problem that remains is to find the value of α'' from equation (12) of Chapter V, to the desired accuracy. The computer can be programmed very easily to serve this purpose. The first two terms of the equation are very small and so can be neglected for the time being. Writing the equation as

$$\alpha''^3 + \beta_c''\alpha''^2 + \frac{E_a^*}{E^*} q\,\alpha'' + \beta_c''\,q = 0 \tag{1}$$

where

$$q = \frac{L''^2}{1 + K'\,10^{-5} + KK'\,\frac{E_a^*}{G}\,10^{-5}}$$

Equation (1) will have three roots. One of them will be a negative real quantity and the other two will be a pair of complex conjugate quantities with a negative real part. The first root, that is the negative real quantity, will not cause vibrations as in that case the beam would be overdamped. The negative real quantity in the other two roots determines the amount of damping.

Suppose the negative root is $(-\alpha_1'')$ and H is a small quantity. Then

$$-\alpha_1'' = -\,(\beta_c'' - H)$$

An attempt is made to evaluate H as follows:

Factorizing equation (1) yields

$$(\alpha'' + \alpha_1'')(\alpha''^2 + H\alpha'' + M) = 0$$

This factorization may be accomplished by writing the coefficients in the following manner:

$$\begin{array}{cccc}
1 & \beta_c'' & \dfrac{E_a^*}{E^*}\,q & \beta_c'' \\
 & \alpha_1'' & H\,\alpha_1'' & M\alpha_1'' \\
\hline
1 & H & M & 0
\end{array}$$

Thus the factors are

$$(\alpha'' + \alpha_1'')$$

and

$$(\alpha''^2 + H\,\alpha'' + M)$$

If the estimated value of H is H_1 where H_1 is greater than H, then the remainder in the last column will be a positive quantity, and not zero. That is

$$[S]_{\alpha''=\beta_c''-H_1} = \alpha''^3 + \beta_c''\,\alpha''^2 + \frac{E_a^*}{E^*}\,q\,\alpha'' + \beta_c''\,q > 0$$

Similarly, if the estimated value of H is H_2 such that H_2 is less than H, then

$$[S]_{\alpha''=\beta_c''-H_2} = \alpha''^3 + \beta_c''\alpha''^2 + \frac{E_a^*}{E^*} q\,\alpha'' + \beta_c'' q < 0$$

Thus the value of H is adjusted so that

$$[S]_{\alpha''=\beta_c''-H}$$

becomes zero.

The logarithmic decrement coefficient will be the real part of the root of the equation

$$\alpha''^2 + H\alpha'' + M = 0$$

or

$$\delta'' = -\frac{H}{2}$$

$$= -\frac{\beta_c'' - \alpha_1''}{2}$$

Thus, the theory of computer solution is to find a value of H such that

$$[S]_{\alpha''=\beta_c''-H} = 0$$

and then

$$\delta'' = -\frac{\beta_c'' - \alpha_1''}{2}$$

Figures 8 and 9 show the results of computer solution for the roots of equation (11), Chapter (V). Families of curves for constant L'', (nondimensional length) are shown in Figure 8, and the same relation yields families of curves for constant β_c'' (nondimensional viscosity parameter) shown plotted in Figure 9. Thus, the data is presented in two different forms such that the effect of coating frequency on the logarithmic decrement for a constant L" can be depicted and alternately the effect of changing material length on logarithmic decrement can be depicted. The roots of equation (11), Chapter (V) are shown plotted in Figure 8 and Figure 9 to a log-log scale as an expedient to summarize the data.

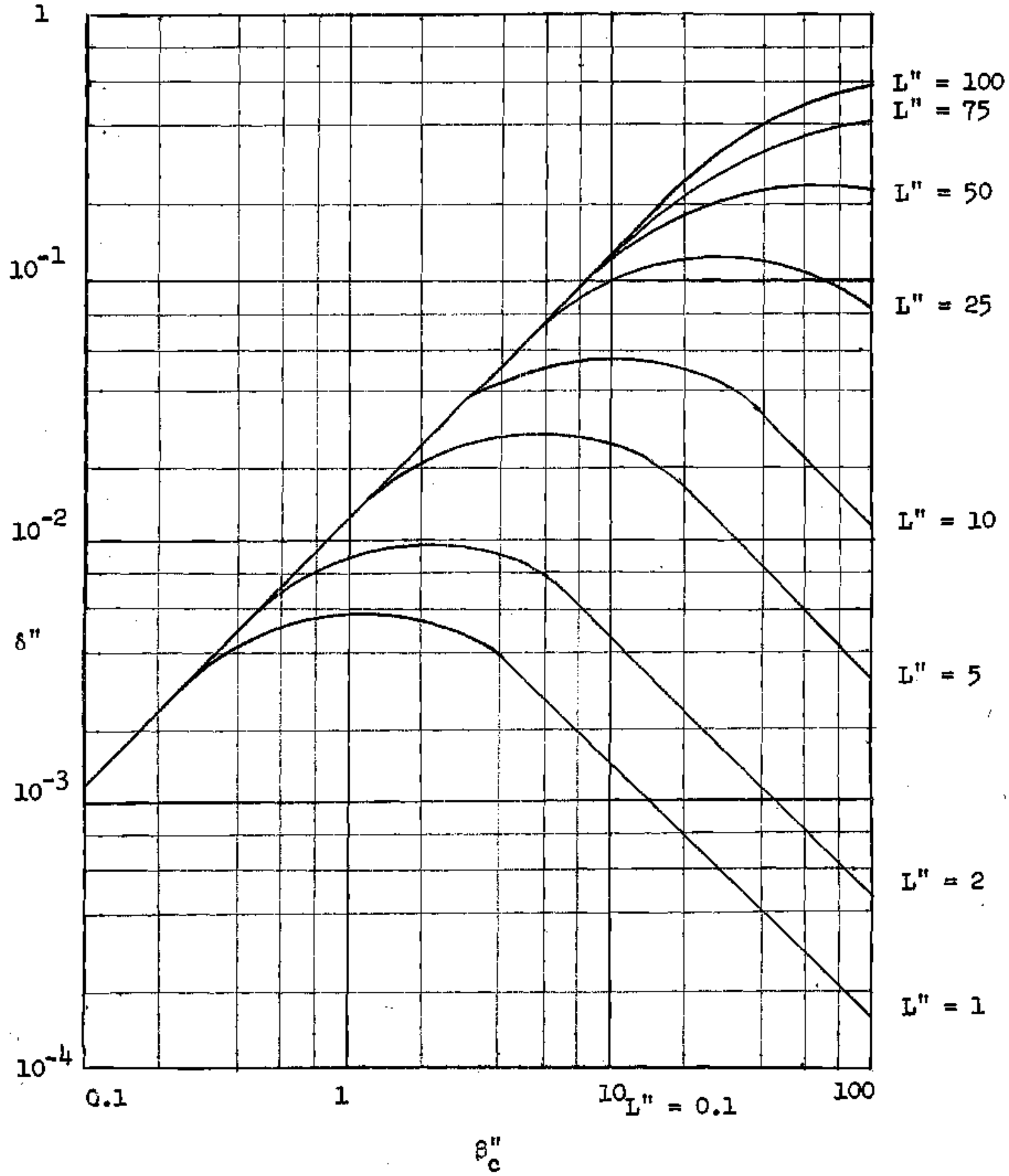


Figure 8. Logarithmic Decrement Constant L"

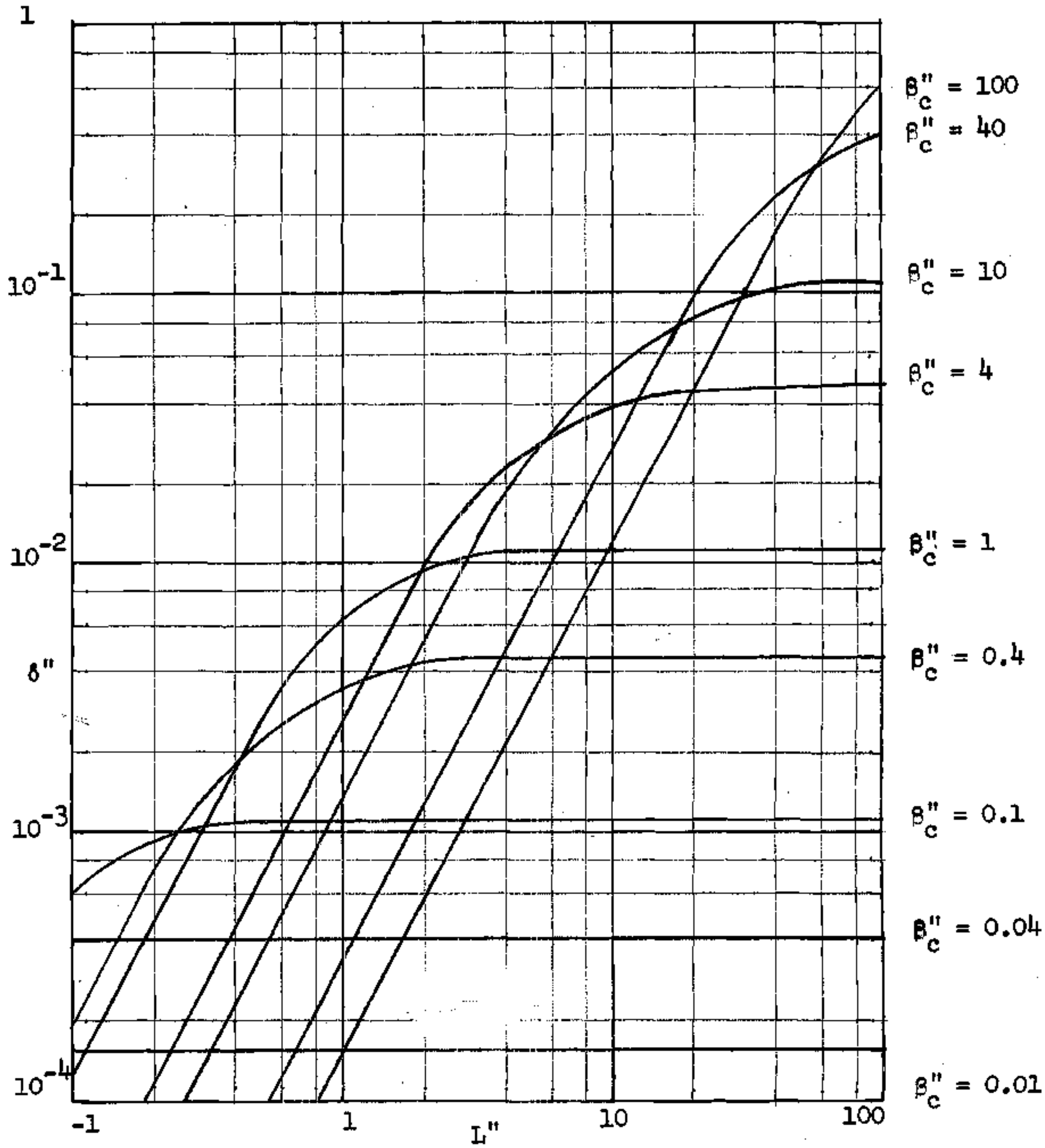


Figure 9. Logarithmic Decrement Constant β_c''

CHAPTER VII

EQUIPMENT AND INSTRUMENTATION

The equipment used in this study may be classified into two categories:

(a) Equipment for finding dynamic and isothermal modulii of elasticity of the viscoelastic material.

(b) Equipment for recording the damped vibration response of the beam.

The equipment used for finding the dynamic and isothermal modulii of elasticity was Instron Tensile Testing Instrument, Type TT. The essential features required of the instrument for measuring isothermal modulus of elasticity are:

(a) A means for applying load at infinitely low speed.

(b) A means for recording load and elongation at any desired magnification factor.

If the same piece of equipment is to be used for finding dynamic elastic modulus also, then it should have an additional feature of applying load almost instantaneously and then being capable of recording the same. The Instron Type TT embodies all these characteristics.

The equipment used for recording the damped vibration response of the beam was a Tektronix Oscilloscope. The oscilloscope used had the following specifications.

Type 502 - Dual Beam

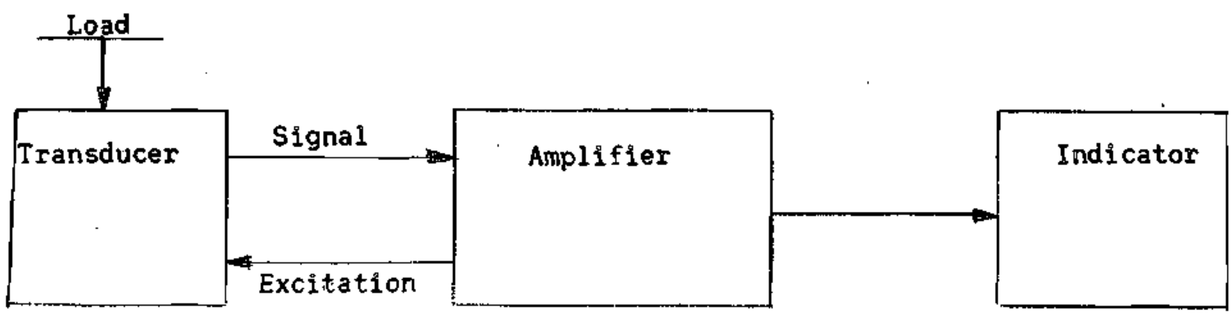


Figure 10. Transducer-Preamplifier Block Diagram.

Sensitivity - 20 volts to 200 μ volts/cm

Sweep - 5 secs. to 1 μ sec/cm

The ballast circuit consisting of four strain gages of the specifications:

Type A-7

Res: 120.5 ± .3 Ω

Gage factor: 1.93 ± 2%

were used in conjunction with a 45v battery. Figure 11a shows the ballast circuit, and Figure 11b shows the actual arrangement of the system.

The experiments were carried out on several cantilever beams of different proportions. Each beam was held vertically in a vise. The vise was mounted on a heavy table and the ratio of the weight of the vise-table system, and the weight of the vibrating beam was of the order of 1000. As no beat phenomena were observed in the response of the beam, it may be assumed that the vise was rigid. The two active strain gages of the ballast circuit were mounted on the beam as close to the vise as possible. The other two strain gages were mounted on a steel plate and enclosed in a shielded box. All connections were made with shielded wires. The beam was given a small initial displacement by hand and then allowed to vibrate freely. The time and sensitivity controls on the oscilloscope were adjusted till the response was obtained on the full range of the oscilloscope screen. The response on the oscilloscope screen was on displacement - time axes. A polaroid camera was mounted on the oscilloscope to record the response.

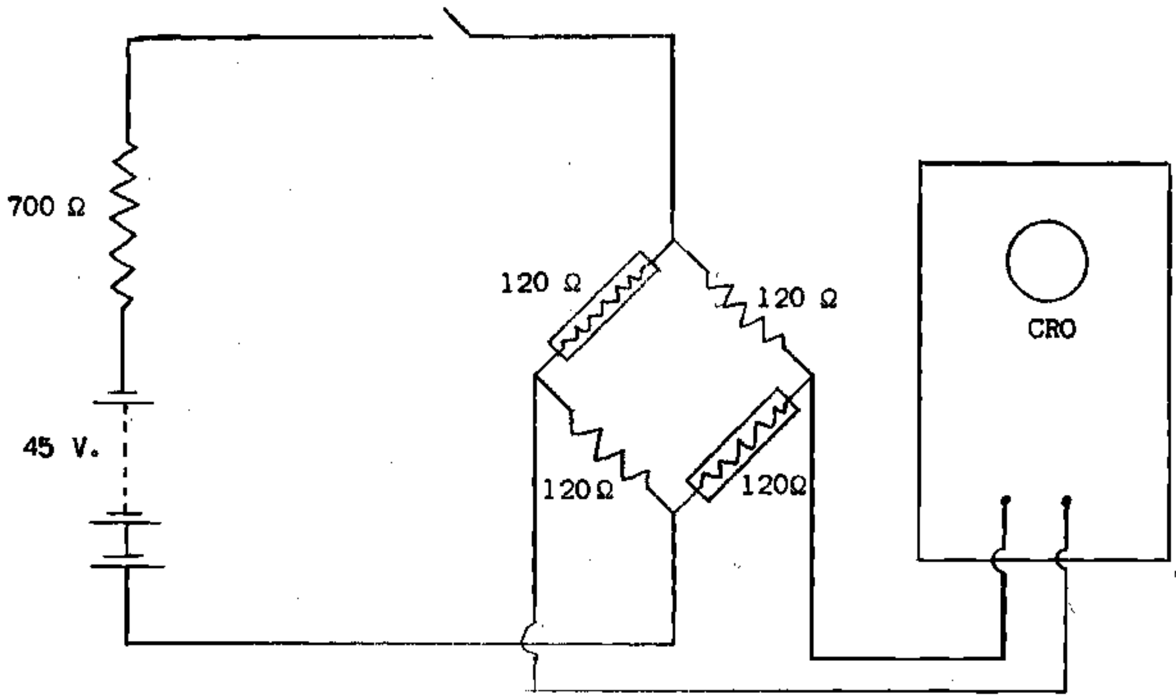
Investigators have found that the Kelvin-Maxwell combination model best explains the behavior of the viscoelastic material. This assumes that the damping is linear and that the energy considerations need not be carried out. Their investigation and the assumptions have been accepted in the development of the equation of motion. For the verification purposes, only the logarithmic decrement was found experimentally. In computing the logarithmic decrement, since the logarithmic decrement is independent of the amplitude, any two consecutive oscillations were used. The logarithmic decrement was computed from n^{th} and $(n+1)^{th}$ oscillations and p^{th} and $(p+1)^{th}$ oscillations, and then the mean taken. In most of the cases, n was the first, and p, the second oscillation.

The beams used in the experiments ranged in the dimensions as follows:

length	3 to 11 in.
thickness	0.015 to 0.050 in.
width	2 in.

The equation of the motion was developed on the assumption that the amplitude of the vibration was small. It is being assumed that with the small amplitudes the bond is not being effected. However, no attempt was made to find to what extent the bond was effected or to study the age hardening.

It was also found that two beams with different coating materials but made to same L'', β_c'', and C^* gave nearly the same logarithmic decrement. As the experimental results correspond very well to the theoretical derivations, it may be assumed that the assumptions made are correct and that the factors which have not been considered do not have appreciable effect on the logarithmic decrement.



(a)

Circuit for Oscilloscope

(b)

Schematic Diagram of the Circuit

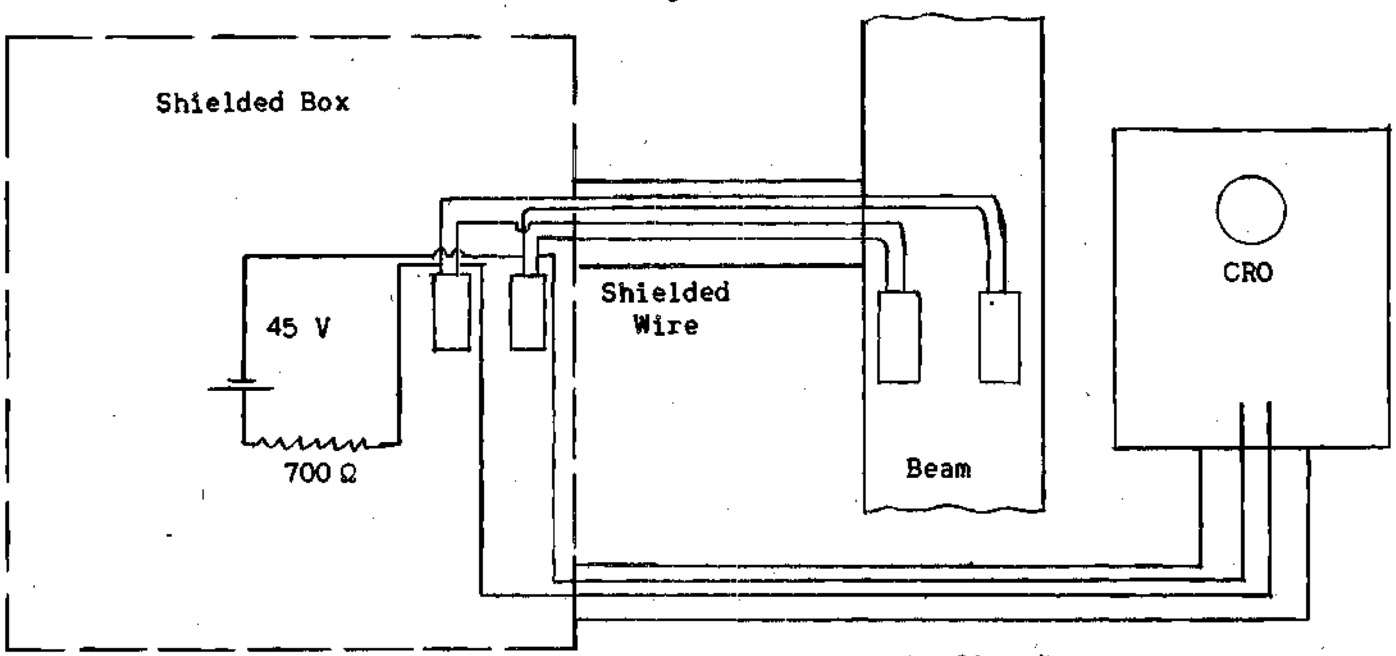


Figure 11. Schematic Diagram of the Circuit.

CHAPTER VIII

DISCUSSION OF THE RESULTS

A constant value of C^* results in a family of curves where variables are δ'', β_c'' and L''.

Figure 8 shows a family of curves for

$$C^* = 1.025$$

where

$$C^* = \frac{E_s + (E_c)_A \cdot \frac{I_c}{I_b}}{E_s + (E_c) \cdot \frac{I_c}{I_b}}$$

$$= \frac{E_s + (E_c)_A \cdot (\frac{d}{D})^3}{E_s + (E_c) \cdot (\frac{d}{D})^3}$$

For any set of beam and coating materials, the thickness ratio,

$$(\frac{d}{D})$$

is defined by this equation.

$$L'' = P \cdot K' \cdot \frac{\pi^2 D^2}{l^2} \cdot 10^5$$

$$\beta_c'' = P \sqrt{\frac{E_c I_c}{\rho_c l^4}} \cdot \frac{D}{Co^*} \cdot 10^5$$

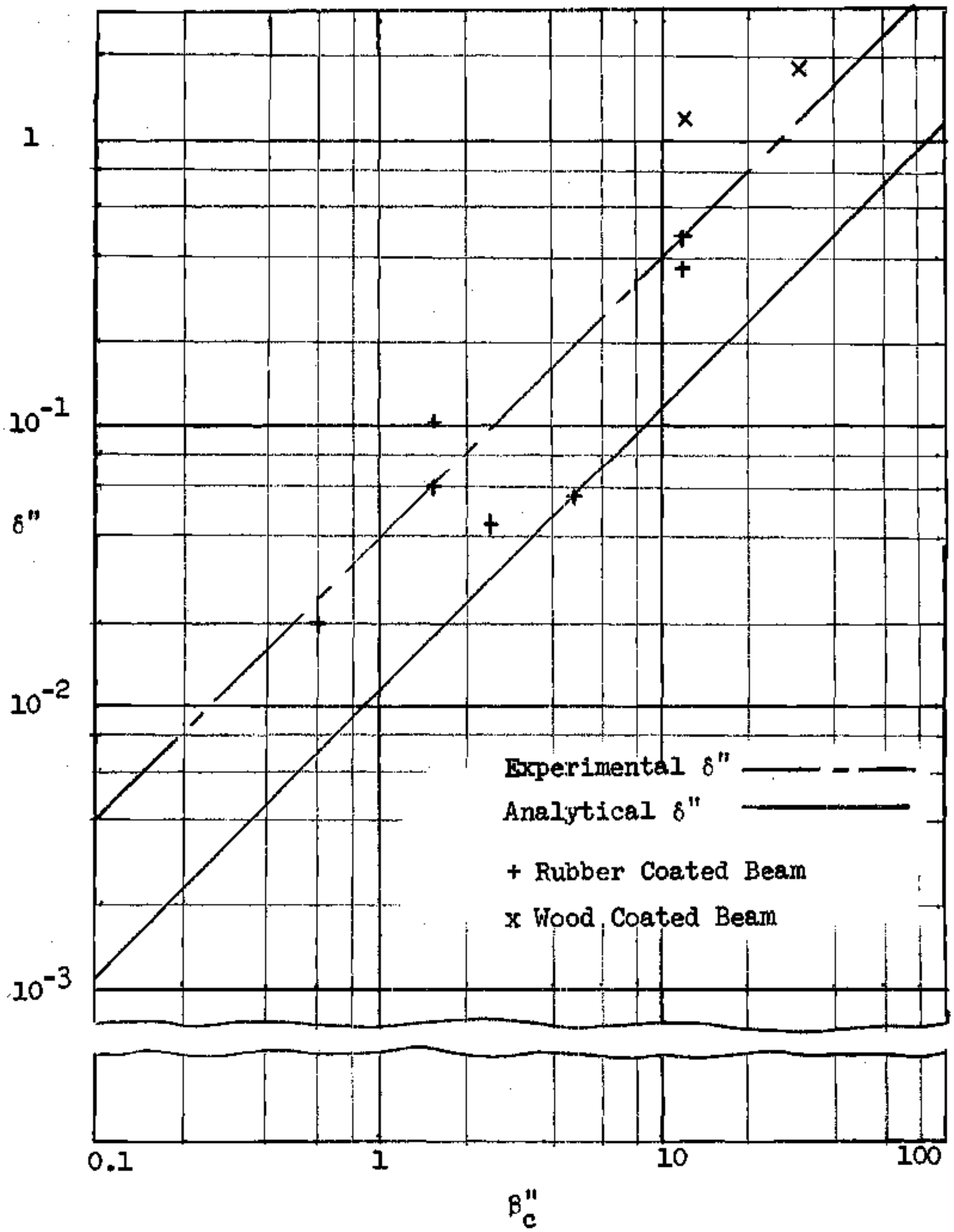


Figure 12. Logarithmic Decrement

where P and R are constants depending upon the manner in which the beam is supported, i.e. simply supported, or cantilever etc.

For epoxy resin as coating material and steel as the cantilever beam material,

$$\beta_c'' = 0.92\, L''$$

The corresponding relationships for wood and rubber as coating materials are

$$\beta_c'' = 0.42\, L''$$

and

$$\beta_c'' = 0.48\; L'' \quad \text{respectively}$$

Thus from Figure 8, it is not possible to have a beam-coating arrangement which will give results near the peak of the curve, or on its right side. The peak occurs when

$$\beta_c'' = L''$$

A coating material of high modulus of elasticity, and very low density will give results near the peak, or on its right side. Laboratory equipment limitations have hampered the study of such a material.

For lower values of β_c'', the experimental curve is of the same nature as that obtained theoretically, except that the slope of the curve is slightly different from that of the theoretical curve. Figure 12 shows the experimental values.

Effect of Changing the Value of C^*

From Chapter (VI), for M_2 to be zero,

$$
\begin{array}{cccc}
1 & \beta_c'' & C^*L^2 & \beta_c'' L''^2 \\
 & -(\beta_c'' - H_1) & -H(\beta_c'' - H_1) & -(\beta_c'' - H_1)\, C^*L''^2 - H_1(\beta_c'' - H_1) \\
\hline
1 & H_1 & & \beta_c''L''^2 - (\beta_c'' - H_1)C^*L''^2 - H_1(\beta_c'' - H_1)
\end{array}
$$

$$\beta_c''L''^2 - (\beta_c'' - H_1) \cdot C^*L''^2 - H(\beta_c'' - H) = 0$$

and the substitution

$$H_1 = -2\delta''$$

$$\delta''^3 + \delta''^2\beta_c'' + \frac{\beta_c'' + C^* L''^2}{4}\,\delta'' + \frac{\beta_c''L''^2(C^* - 1)}{8} = 0$$

For

$$\beta_c'' \geq 0$$

$$L'' \geq 0$$

and

$$C^* \geq 1$$

the equation has one real negative root, and a pair of complex conjugate roots. Only the real negative part of the complex root has the physical significance of logarithmic decrement. Figure 13 shows the plot of β_c'' and δ'' for a fixed L'' and different C^*. Figure 14 is the same plot on log-log paper.

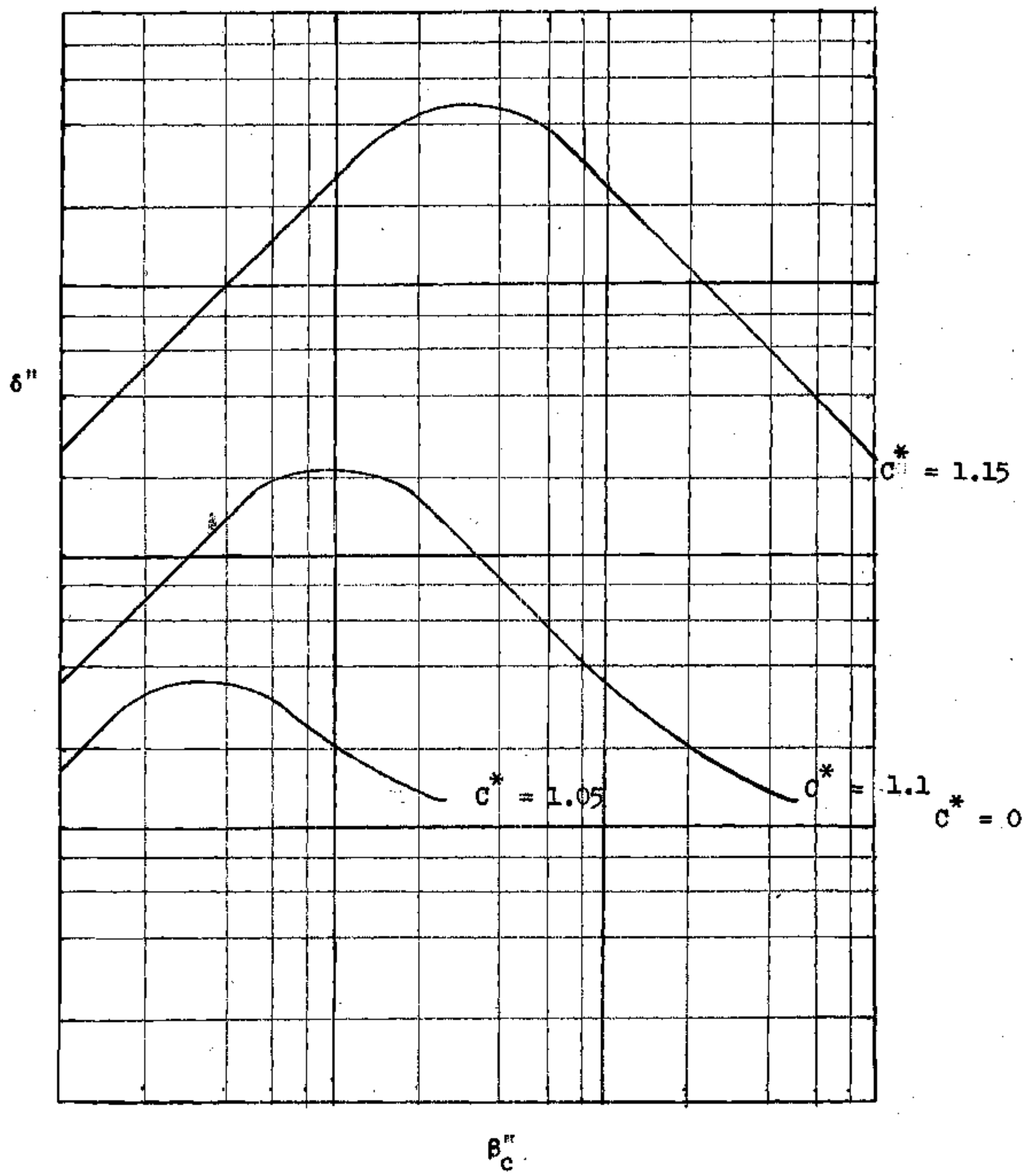


Figure 13. Logarithmic Decrement
Constant L", Variable C^*

From the equation above and Figure 14, it is seen that for the value of C^* of unity, the logarithmic decrement is identically zero. This is only reasonable since this case is defined as that of equal dynamic and isothermal modulii of elasticity. There is no viscoelastic effect in the coating. The other curves for varying C^* are seen to have parallel slope on the rising portion. This fact is borne out in Figure 14, which is plotted from a computer solution of the equation for varying values of C^*.

The maximum value of the logarithmic decrement is found from the equation

$$\delta''^3 + \delta''^2 \beta_c'' + \frac{\beta_c''^2 + C^* L''^2}{4} \delta'' + \frac{\beta_c''^2 L''^2 (C^* - 1)}{8} = 0$$

$$\frac{d\delta''}{d\beta_c''} = 0$$

when

$$\delta''^2 + \frac{\beta_c''}{2} \delta'' + \frac{L''^2 (C^* - 1)}{8} = 0$$

or when

$$\delta''_{max} = \frac{-\beta_c'' + \sqrt{\beta_c''^2 - 2L''^2 (C^* - 1)}}{4}$$

It is seen from Figure 8 and Figure 9, that theoretically there exists a maximum value of logarithmic decrement which a coated beam can exhibit. Further increase in β_c'' beyond that value at δ''_{max} results in decreasing values of logarithmic decrement. One recalls that it was

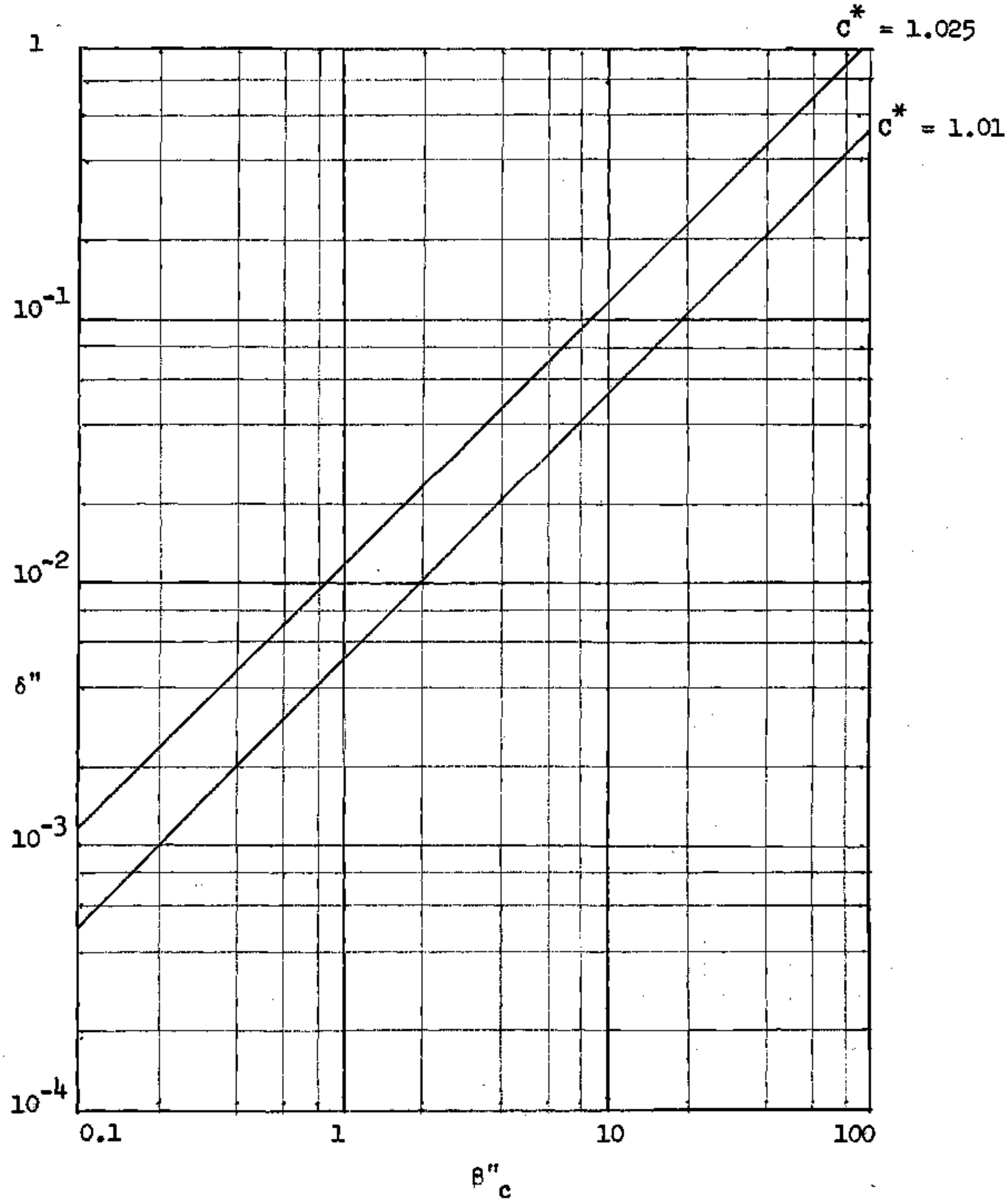


Figure 14. Envelope of L"

(a)

(b)

Figure 15. Coated Beam.

shown that β_c'' is a function of L'' for a given system. It is observed from Figure 8 for $C^* = 1.025$ that the maximum logarithmic decrement occurs when β_c'' is numerically equal to L''.

It was found that the effects of rotary inertia and shear in the beam were so small that they need not be considered for the design purposes. However, consideration of these effects assumes importance because they give a more exact or nearly exact equation of motion. One can compare columns 5 and 6 of Table 3. Column 5 is the value of the equation including inertia and shear terms. Column 6 is the value of the equation with shear and inertia terms ignored.

It was also observed that the damping effects were considerably less in the system in Figure 15a than in the system shown in Figure 15b.

Recommendations

It is recommended that viscoelastic materials with low density and high modulus of elasticity be studied for their properties and then used as the damping coating. This may result in obtaining data at the peak or on the right side of the peak in Figure 8. Styrafoams are likely to have such properties.

A ballast circuit was used with an Oscilloscope to obtain the response of the vibrating beam. The Sanborn recorder may be used along with the oscilloscope to obtain the same effects.

It was assumed that the damping effect of the adhesive binding the viscoelastic material on the beam, was negligible as compared to the damping effect of the viscoelastic layer. It is recommended that the equation of motion be expanded to include the damping effect of the adhesive also. An alternative to this is to evaluate the dynamic and isothermal modulii of elasticity, and the density of the adhesive. The designer can then choose an adhesive having properties similar to the coating material. In this investigation, a butyl based rubber adhesive was used to bond the rubber coating to the beam.

One recalls that the effect of internal damping in the steel beam was not included in the derivation. For comparison purposes, it is worthwhile to run experiment in vacuum to see the damping effects of the surrounding air.

A P P E N D I X

Table 1

Values of P and R

Beam	P	P	R
Simply Supported	$\frac{\pi^2}{\pi^2}$	1.000	1.571
Cantilever	$\frac{(1.875)^2}{\pi^2}$	0.356	0.555
Clamped-Simply Supported	$\frac{(3.927)^2}{\pi^2}$	1.565	2.450
Clamped-Clamped	$\frac{(4.73)^2}{\pi^2}$	2.260	3.555
Free-Free	$\frac{0}{\pi^2}$	0.000	0.000

Table 2

Experimental Results for

$C^{*} = 1.025$

β_c"	L"	δ"
0.8	1.6	0.020
1.5	3.1	0.110
1.5	3.1	0.063
2.5	5.0	0.048
4.2	8.6	0.052
11.6	24.0	0.520
11.6	24.0	0.350
11.6	24.0	1.270
33.3	24.0	1.900

Table 3

Computer Results for $C^* = 1.025$

L''	β_c''	α''	δ''	S1	S
100.0	1.0	-.97561	-.012194	.2321,-06	.0000, 00
100.0	.1	-.09756	-.001220	.1010,-01	.1010,-01
75.0	100.0	-99.09774	-.451130	.2212, 00	.1300, 00
75.0	10.0	-9.76006	-.119970	.1122,-01	.1100,-01
50.0	40.0	-39.39232	-.303840	.1594, 00	.1500, 00
50.0	4.0	-3.90301	-.048492	.9014,-02	.9000,-02
25.0	40.0	-39.71823	-.140882	.4465,-02	.0000, 00
25.0	.1	-.09756	-.001220	.6480,-03	.6480,-03
10.0	100.0	-99.97524	-.012380	.1054,-01	.8000,-02
10.0	.1	-.09756	-.001220	.1240,-03	.1240,-03
5.0	40.0	-39.98460	-.007700	.1694,-01	.1670,-01
5.0	.1	-.09756	-.001220	.4820,-04	.4820,-04
.5	100.0	-99.99993	-.000035	.7502,-01	.7501,-01
.5	40.0	-39.99984	-.000080	.6043,-02	.6041,-02
.1	100.0	-99.99999	-.000005	.7500,-01	.7500,-01

Sample Calculations

Suppose a cantilever steel beam of length 8" and thickness 1/18 in., and coated with rubber of thickness 1 in. gives a logarithmic decrement

$$\delta' = \ln \frac{X_2}{X_1} = 0.12$$

$$C^* = \frac{E_b + (E_c)_a \cdot \frac{I_c}{I_b}}{E_b + (E_c) \cdot \frac{I_c}{I_b}}$$

$$= \frac{30 \times 10^6 + 1400 \times 18^3}{30 \times 10^6 + 1200 \times 18^3} = 1.025$$

$$C_o^* = \left[\frac{E_b + E_c \cdot \frac{I_c}{I_b}}{\rho_b + \rho_c \frac{A_c}{A_b}}\right]^{1/2}$$

$$= \left[\frac{30 \times 10^6 + 1200 \times 18^3}{7.5 \times 10^{-4} + 0.6 \times 10^{-4} \times 18}\right]^{1/2}$$

$$= 1.5 \times 10^5$$

$$L'' = P \cdot K' \cdot \frac{\pi^2 D^2}{l^2} \cdot 10^5$$

$$P = 0.356 \quad \text{(Table 1)}$$

$$K' = \sqrt{\frac{I_b}{AD^2}} = \sqrt{\frac{1}{12}}$$

$$L'' = 0.356 \cdot \sqrt{\frac{1}{12}} \cdot \frac{\pi^2}{18^2 \times 8^2} \cdot 10^5 = 5.0$$

$$\beta_c'' = P \cdot \sqrt{\frac{E_c}{\rho \cdot 12}} \cdot \frac{D\ d}{l^2\ C_o^*} \cdot 10^5\ \pi^2$$

$$= 0.356 \cdot \sqrt{\frac{1200}{0.6 \times 10^{-4} \times 12}} \cdot \frac{10^5}{18 \times 8^2 \times 1.5 \times 10^5} \cdot \pi^2$$

$$= 2.5$$

$$\delta'' = \delta' \cdot \frac{D}{C_o^*} \cdot 10^5 \cdot \omega$$

$$= 0.12 \cdot \frac{10^5}{18 \times 1.5 \times 10^5} \cdot 0.555 \cdot \frac{1.5 \times 10^5}{18 \times 8^2} \cdot \frac{1}{2\pi}$$

$$= 0.045$$

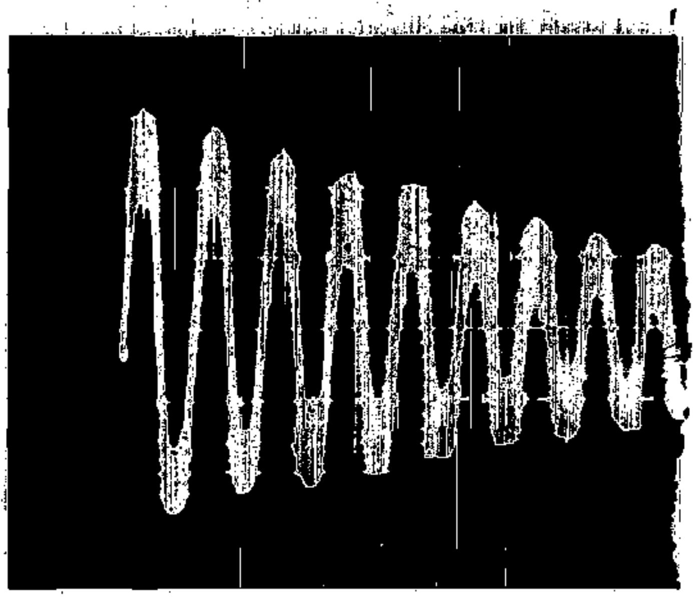


Figure 16. Response of 8" Cantilever Beam.

Time Scale 1 cm = 5 m.s.

```
0200     BAC-220  STANDARD VERSION   2/1/62                                    $
0200  COMMENT SOLUTION OF EQN (12)CHAPTER 5                                    $
0200  REAL    BC,H,S,DEL,A,L,S1,S2                                             $
0200  INTEGER I                                                                $
0200  INPUT   MAT (C)                                                          $
0207  INPUT   DOG (N)                                                          $
0214  INPUT   CAT (L)                                                          $
0221  OUTPUT  RAT (L,BC,A,DEL,S1,S)                                            $
0243  OUTPUT  PAT(C)                                                           $
0250  FORMAT  FM(B5,X7.3,B5,X6.2,B5,X9.5,B5,X8.6,B5,F9.4,B5,F9.4,B5,F9.4,W0) $
0267  FORMAT  FMT ( B6, * C = *,X7.4,W2)                                       $
0274  FORMAT  FMT1(B8,*L*,B10,*BC*,B11,*A*,B12,*DEL*,B10,*S1*,B12,*S*,W2)     $
0289          READ ($$MAT)                                                     $
0293          WRITE ($$PAT,FMT)                                                $
0301          WRITE ($$FMT1)                                                   $
0305          READ ($$ DOG)                                                    $
0309          I = 1                                                            $
0311  L1..    READ ($$ CAT)                                                    $
0315          BC = 100.00                                                      $
0317  L2..    H = BC/100                                                       $
0321  L3..    A = (-(BC-H))                                                    $
0326          S = ((A)*3 + (BC).((A)*2) +   (C).(A).((L)*2)+(BC).((L)*2))     $
0347          IF ABS (S) LSS ((0.00001).BC)                                    $
0347          GO TO L6
0356          IF S GTR 0                                                       $
```

```
0356  BEGIN
0356         H = H/10                                                      $
0363         IF ABS (H) LSS 0.00001                                        $
0363         GO TO L6                                                      $
0369         GO TO L3                                                      $
0370  END                                                                  $
0370  L4..   A = A + H                                                     $
0373         S = ((A)*3 + (BC).((A)*2)+  (C).(A).((L)*2)+(BC).((L)*2))     $
0394         IF ABS (S) LSS ((0.00001).BC)                                 $
0394         GO TO L6                                                      $
0403         IF S LSS 0                                                    $
0403         GO TO L4                                                      $
0407         H = -H/10                                                     $
0411         IF ABS (H) LSS 0.00001                                        $
0411         GO TO L6                                                      $
0417  L5..   A = A + H                                                     $
0420         S = ((A)*3 + (BC).((A)*2)+  (C).(A).((L)*2)+(BC).((L)*2))     $
0441         IF S GTR 0                                                    $
0441         GO TO L5                                                      $
0445         H = -H/10                                                     $
0449         IF ABS (S) LSS ((0.00001).BC)                                 $
0449         GO TO L6                                                      $
0458         GO TO L4                                                      $
0459  L6..   DEL = -(A+BC)/2                                               $
0466         S1=(A+BC).((3.((A/10)*4)/(10*7)) + ((A*2)/(10*5))) + S        $
```

```
0505          WRITE ($$RAT,FM)                $
0513          IF BC NEQ 0.01                  $
0513   BEGIN
0513          IF BC EQL 0.04                  $
0513          GO TO L7                        $
0523          BC = BC/10                      $
0527          GO TO L2                        $
0528   END                                    $
0528          IF BC EQL 0.01                  $
0528   BEGIN
0528          BC = 40                         $
0535          GO TO L2                        $
0536   END                                    $
0536   L7..   IF I NEQ N                      $
0536   BEGIN
0536          I = I+1                         $
0545          GO TO L1                        $
0546   END                                    $
0546          STOP                            $
0547          FINISH                          $

COMPILED PROGRAM ENDS AT 0548
PROGRAM VARIABLES BEGIN AT 4337
```

GLOSSARY OF ABBREVIATIONS

Note: Suffix b stands for beam and c for the coating

A Area of Cross Section

B Maximum amplitude

$C = \frac{E_a^*}{E^*}$

C_o^* Velocity of longitudinal wave in the beam

D Thickness of the beam

d Thickness of the coating

E Young's Modulus

$E_a^* = E_b + (E_c)_A \cdot \frac{I_c}{I_b}$

$E^* = E_b + E_c \cdot \frac{I_c}{I_b}$

G Shear Modulus

I Moment of Inertia (Area)

K Constant depending upon the section

$K'^2 = \frac{I}{AD^2}$

$L' = \frac{\pi D}{l}$

$L'' = K' \cdot \frac{\pi^2 D^2}{l^2} \cdot 10^5$

l Length of the beam

M Moment

P Constant depending upon the beam

R Constant depending upon the beam

Q Shear Force

$t' = t \cdot \frac{C_o^*}{D}$

w Displacement in Y direction

$w' = \frac{w}{D}$

$x' = \frac{x}{D}$

Z Distance of the element from the neutral axis

$\alpha'' = \alpha' \cdot 10^5$

β_c Viscosity parameter of the coating

$\beta_c' = \frac{D\beta_c}{C_o^*}$

$\beta_c'' = \frac{D\beta_c}{C_o^*} \cdot 10^5$

β Angle of shear

σ Stress

ε Deformation

θ Slope of the deflection curve

$\rho = \rho_b + \rho_c \frac{A_c}{A_b}$

LITERATURE CITED

1. R. A. Anderson, "Flexural Vibrations in Uniform Beams According to the Timoshenko Theory," Journal of Applied Mechanics 1953, pp. 504.

2. Unger, Ross and Kervin, "Damping of Flexural Vibrations by Alternate Viscoelastic and Elastic Layers," WADC Technical Report 59-509.

3. A. Vanitterbeek and H. Myncke, "Vibrations of Plates Covered with a Damping Layer," Acustica, Vol. 8, 1958.

4. A. W. Nolle, "Dynamic Mechanical Properties of Rubber Like Materials," Journal of Polymer Science, Vol. 5, 1950, pp. 1-54.

5. F. Schwarzl, "Forced Bending and Extensional Vibration of a Two-layer Compound Viscoelastic Beam," Acustica, Vol. 8, pp. 164.

6. J. A. Whittier, "The Effect of Configurational Additions Using Viscoelastic Interfaces on the Damping of a Cantilever Beam," WADC Technical Report 58-568, ASTIA Document No. 214381.

7. H. J. Plass, "Damping of Vibration in Elastic Rods and Sandwich Structures by Incorporations of Additional Viscoelastic Material," Third Midwestern Conference on Solid Mechanics at Univ. of Texas, 1957.

8. Timoshenko, "Vibration Problems in Engineering," Nostrand Co. Inc., 1937.

9. W. Nowacki, Dynamics of Elastic Systems, John Wiley and Sons Inc., 1963.

10. B. A. Boley and C. C. Chao, "Some Solutions of the Timoshenko Beam Equation," Journal of Applied Mechanics, 1955.

11. Julius Miklowitz, "Flexural Wave Solutions of Coupled Equation Representing the More Exact Theory of Bending," Journal of Applied Mechanics, 1953, pp. 511.

12. R. P. Thorn, "Built-in Damping," Machine Design, Nov. 25, 1965, pp. 174.

13. P. Lienard, "Les Mesures D'Amortissment Dans Les Materiaux Plastiques Ou Fibreux," Cahiers D'Acoustique, No. 79, 1957, pp. 359.

14. Von Hermann Oberst, "Uber Die Dampfung Der Biegeschwingungen Dunner Bleche Durch Fest Haftende Belage" Acustica, 1952.

15. Washington University Engineering Experiment Station, Bulletin No. 114, 1947.

16. D. P. Giddens, "Investigation of the Effect of Various Damping Materials on Dynamic Stresses in Flat Panels," Lockheed-Georgia Co., SRD 72-21-419.

17. J. S. Whittier, "Rheological Properties of Adhesives Considered for Interface Damping," WADC-TR 60-280.